Physics 432/750: Cosmology
Winter 2003-2004
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Problem Set 1 Solutions

1. Robertson-Walker metric

Show that the Robertson-Walker metric

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t)[dr^2 + S_k^2(r) d\psi^2]$$

can also be written in the form

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t)[dr^2/(1 - kr^2) + r^2 d\psi^2]$$

A quick solution is to work backwards for the second form. For r in the second equation, substitute

$$r = \sin r', r', \mathrm{or} \sinh r'$$

for the cases $k = +1, 0, -1$, respectively. The derivatives of this are

$$dr = \cos r' dr', dr', \cosh r' dr'$$

and the first term within brackets becomes

$$\frac{dr^2}{1 - kr^2} = dr'^2$$

for all three cases. Substitution into the second form yields the first upon the trivial replacement of r' with r and identification of $S_k(r)$ with the three cases.

2. Metric for an Open Universe

For a $k = -1$ Friedmann cosmology ($\Lambda = 0$), with $\rho = p = 0$, show that the R-W metric line element becomes

$$c^2 d\tau^2 = c^2 dt^2 - c^2 t^2 [dr^2 + \sinh^2 r d\psi^2]$$

For $\rho = p = 0$ and $k = -1$, the Friedmann equation is

$$\dot{R}^2 = c^2$$

which implies that $R = ct$. Substitution into the R-W line element yields

$$c^2 d\tau^2 = c^2 dt^2 - c^2 t^2 [dr^2 + \sinh^2 r d\psi^2]$$

3. Relativistic Velocities

(a) Show that the general relativistic relation between recession velocity and cosmological redshift is

$$v_{rec}(t, z) = \frac{c}{R_0} \dot{R}(t) \int_0^z \frac{dz'}{H(z')}$$

where $H(z')$ is the Hubble constant at redshift z'.

The recession velocity is the time derivative of the proper distance. For RW metric,

$$ds^2 = -c^2dt^2 + R^2(t)[dr^2 + S_k(r)d\psi^2]$$

where r is the dimensionless comoving radial coordinate. The proper distance to an object is along a surface of constant time, $dt = 0$, thus $ds = Rdr$. Integrating over the line of sight yields the time-dependent proper distance

$$D(t) = R(t)r$$

The recession velocity is therefore

$$v(t, z) = \dot{R}(t)r(z) = H(t)D(t)$$

which is Hubble's law. Now, what is $r(z)$? Use the metric again. When we observe an object, we see a photon that travelled along the line of sight, which is a null geodesic and has $ds = 0$ and $d\psi = 0$. Thus

$$cdt = R(t)dr$$

The comoving coordinate of an object seen from photons that it emitted at time t_{emit} is

$$r(t_{emit}) = c\int_{t_{emit}}^{t_0} \frac{dt'}{R(t')}$$

The scale factor is related to redshift by

$$1 + z = \frac{R_0}{R(t)}$$

thus

$$\frac{dt}{R(t)} = -\frac{dz}{R_0 H(z)}$$

where we substituted $H(z) = \dot{R}/R$. Now put this all together,

$$r(z) = \frac{c}{R_0}\int_0^z \frac{dz'}{H(z')}$$

and multiply by $\dot{R}(t)$ to get the relation between proper distance and redshift. Note carefully that the general relativistic recession velocity depends on both redshift and time. At fixed t_0, the redshift z uniquely specifies the comoving coordinate $r(z)$ from the observer to the object. The recession velocity of that object depends on the time t, through the varying rate of change of the scale factor, $R(t)$. If you want to know "at what rate is the object receding from us now?" then you want $v(z, t_0)$. If you want to know, "at what rate was the object receding when it emitted the light we now observe?" then you want $v(z, t_{emit})$.

(b) Show that the special relativistic relation between peculiar velocity and redshift is

$$v_{pec}(z) = c\frac{(1+z)^2 - 1}{(1+z)^2 + 1}$$

Given the relativistic Doppler formula

$$1 + z = \sqrt{\frac{1 + v/c}{1 - v/c}}$$

you can algebraically solve for this form.

(c) Show that both relativistic relations are approximately $v \approx cz$ at small distance.

At small z, the Hubble constant is "constant," thus integral in the general relativistic expression is simply

$$v_{rec} = \frac{c}{R_0}\dot{R}_0\frac{z}{H_0} = cz$$

because $H_0 \equiv \dot{R}_0/R_0$.

For the special relativistic case, expand the $(1+z)^2$ terms, keeping only the leading order terms in the numerator and denominator yields,

$$v(z) = c\frac{2z}{2} = cz$$

Because the general and special relativistic relations have the same low-redshift approximation some (otherwise very smart) people have mistakenly used the special relavistic form to interpret cosmological redshifts. This is wrong; cosmological recession velocities can be larger than the speed of light. No object is moving at superluminal speed through a Lorentz frame and no information exceeds the speed of light.

4. Particle Horizon

Following the suggestions outlined on pp. 85-86 of the text, show that the dominant form of mass-energy at early times must scale as $\rho \propto R^{-\alpha}$ with $\alpha > 2$ for a particle horizon to exist.

A photon follows a null geodesic, i.e., $d\tau = 0$, therefore the RW metric yields

$$c^2 dt^2 = R^2 dr^2$$

The coordinate distance r that a photon travels is then

$$r = \int_{t_0}^{t_1} \frac{cdt}{R(t)}$$

We can rewrite this integral by making the trivial substition

$$dt = dR\frac{dt}{dR} = \frac{dR}{\dot{R}}$$

The Friedmann equation at early times behaves like that of a flat universe, with $k = 0$,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho$$

which yields

$$\dot{R} \propto \sqrt{\rho R^2}$$

Let the density evolve as a power law $\rho \propto R^{-\alpha}$, and substitute into the equation integral, thus

$$r = \int_{R(t_0)}^{R(t_1)} \frac{dR}{R^{2-\alpha/2}} \propto R^{\alpha/2-1}|_{R(t_0)}^{R(t_1)}$$

To find out how far a photon could travel since the beginning of the universe, we are interested in the limit $t_0 \to 0$, which is the same as $R(t_0) \to 0$. The integral converges if $\alpha \geq 2$, but diverges if $\alpha < 2$. At early times, the universe is radiation dominated and has $\alpha = 4$, the integral does indeed converge and photons travel a finite distance, thus a particle horizon exists (this would also be true even if the universe were matter dominated at early times, because $\alpha = 3$ for matter).

5. Evolution of $\Omega_{matter}, \Omega_{vac}$

Compute how the density parameters Ω_{matter} and Ω_{vac} evolve with time in different cosmologies and plot the results on a figure like 3.5 on p. 83. In other words, for each choice of $\Omega_{matter}, \Omega_{vac}$ today, plot the trajectory of the model backwards in time to where it would lie in that same diagram at $t \approx 0$. On the plot, clearly indicate which end of each curve is now and which is at $t = 0$. Pick models from all regions of the diagram and be sure to include the following (in other words, do more than just these):

$\Omega_{matter} = 1, \Omega_{vac} = 0$

$\Omega_{matter} = 0.2, \Omega_{vac} = 0$

$\Omega_{matter} = 0.3, \Omega_{vac} = 0.7$

The density of the components vary with the scale factor of the universe as $\rho_{rad} \propto R^{-4}, \rho_{matter} \propto R^{-3}, \rho_{vac} = \text{constant}$. Each component $i = \{\text{matter, radiation, vacuum}\}$ of the mass-energy density has a dimensionless density parameter $\Omega_i = \rho_i/\rho_{crit}$ where $\rho_{crit} = 3H^2/8\pi G$. Thus, Ω_i implicitly varies with the scale factor through the dependence of the Hubble parameter on the scale factor. This dependence is shown by a form of the Friedmann equation,

$$H^2(a) = H_0^2[\Omega_{vac} + \Omega_{matter}a^{-3} + \Omega_{rad}^{-4} - (\Omega - 1)a^{-2}]$$

where $\Omega = \Omega_{rad} + \Omega_{vac} + \Omega_{matter}$ and $a = R/R_0$. Given Ω_{vac} and Ω_{matter} today, you can now compute Ω_i as a function of a where $a = 1$ is the present epoch and the limit $a \to 0$ corresponds to $t \to 0$. For a mass-energy component that scales with a as $\rho_i = \rho_{0,i}a^{-\alpha}$, combining the relations above leads to the formula

$$\Omega_i = \frac{\Omega_{0,i}a^{-\alpha}}{[\Omega_{vac} + \Omega_{matter}a^{-3} + \Omega_{rad}a^{-4} - (\Omega - 1)a^{-2}]}$$

where $\Omega_{0,i}$ is the value of the density parameter today. For the purpose of this problem, you can ignore radiation. Now pick some values for $\Omega_{matter}, \Omega_{vac}$ today and draw the curves by slowly varying a from 1 back to near 0.

In my plot (see attached) I put a "0" at $t = \text{now}$ for each curve. Note that ALL of the curves converge to $\Omega_{matter} = 1, \Omega_{vac} = 0$ at $t = 0$. Thus, at early times, all universes look like Einstein-de Sitter. Also note that "flat" models stay flat (evolve along a straight line with $\Omega =$

1, closed models stay closed, and open models stay open, but that acceleration/deceleration (compare to the line where $q_0 = 0$) varies with time. For example, our favorite model, $\Omega_{matter} = 0.3, \Omega_{vac} = 0.7$ began as decelerating and crossed over to accelerating at $z = 0.6$.