Introduction to Lorenz's System of Equations

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1 Preliminaries

1.1 Initial Conditions

This paper discusses some fundamental and interesting properties of the Lorenz equations, a topic which is well outside the scope of a single paper, but has been appropriately narrowed down. It is assumed that the reader is familiar with dynamical systems and bifurcations.

It should be noted that in all of the diagrams, the solutions must be calculated numerically, as analytic solutions are impossible using known methods, so some of the claims are not rigorous. For example, when a "nonperiodic trajectory" is plotted, technically the path is a small stretch of a high period orbit since a computer is a finite machine. Still, the general behavior of the system that is illustrated and the characteristics that emerge do not depend on our method (Sparrow 6).

1.2 Historical Setting

Edward Lorenz formally awoke the scientific world to the idea of deterministic chaos through his 1963 paper, "Deterministic Nonperiodic Flow." Prior to this, a few shrewd minds had identified systems that showed characteristics like nonperiodicity and sensitivity to initial conditions, but the overwhelming mindset was that, outside of the quantum world, classical physics provided the *theory* for completely predicting the state of the universe at any future time[1]. In the mid-20th century, computer and satellite technology were being developed with the ultimate intention of controlling the weather.

The mistake was in believing that tiny perturbations in a system only amount to tiny changes over time. Lorenz showed that such small differences actually amount to drastic changes in a system's behavior. As Gleick (21) puts it, if one infinitely accurate sensor were placed within every cubic foot of the earth's atmosphere, and the data were fed to an infinitely powerful computer, reasonable prediction (e.g. rain vs. shine) would still be limited to less than one month. Prediction becomes suddenly truncated even in a completely deterministic system. Yet the scientific community was reluctant to accept this new idea. Decades later, physicists would commonly nonchalantly cross out small nonlinear terms in order to simplify a system[2]. There was a reluctance to abandon the predictability of the classical universe.

1.3 Derivation

This section provides a brief derivation of the Lorenz equations. The details are not crucial to this paper, as we are more interested in the behavior of the system. For more details regarding this derivation or other derivations, see Kundu, Lorenz, or Sparrow.

In his 1963 paper, "Deterministic Nonperiodic Flow," Lorenz cites the convection equations of Saltzman (1962). These equations come from the examination of a fluid of uniform depth *H*, with a temperature difference between the upper and lower layer of ΔT, in particular with a linear temperature variation. In the case where there is no variation with respect to the *y*-axis, Saltzman provided the governing equations:

$$\frac{\partial}{\partial t}\nabla^2\psi = -\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)} + \nu\nabla^4\psi + g\alpha\frac{\partial\theta}{\partial x} \quad ,$$
$$\frac{\partial}{\partial t}\theta = -\frac{\partial(\psi,\theta)}{\partial(x,z)} + \frac{\Delta T}{H}\frac{\partial\psi}{\partial x} + \kappa\nabla^2\theta \quad ,$$

where ψ is a stream function for the two-dimensional motion, θ is the temperature deviation from the steady state, $\psi, \nabla^2\psi$ vanish at the upper and lower boundaries, and g, α, ν, κ are the respective constants of gravitational acceleration, coefficient of thermal expansion, kinematic viscosity, and thermal conductivity. Rayleigh discovered a critical point at which these equations show convective motion, based on what is now known as the *Rayleigh number*. Lorenz then defined three time dependent variables: *X* proportional to the intensity of the convective motion, *Y* proportional to the temperature difference between ascending and descending currents, and *Z* proportional to distortion of the vertical temperature profile from linearity. The result was the following set of equations:

$$\dot{X} = -\sigma X + \sigma Y$$
$$\dot{Y} = -XZ + rX - Y$$
$$\dot{Z} = XY - bZ \quad ,$$

where a dot denotes the derivative with respect to time, $\sigma = \kappa^{-1}\nu$ is the *Prandtl number*, $r = R_c^{-1}R_a$ (the Rayleigh number over the critical Rayleigh number), and $b = 4(1+a^2)^{-1}$ gives the size of the region approsimated by the system (*a* comes from the solutions for ψ and θ) (Lorenz 134-5). All parameters are taken to be positive. These equations will be henceforth referred to as the Lorenz System.

We will examine the behavior of the Lorenz System for different parameter values to see some of the interesting features, but the system is only a realistic model of the intended fluid convection if *r* is close to 1. However, other authors have discovered many physical problems modeled by essentially the same set of equations, with realistic behavior for a variety of parameter values. Some examples include irregular spiking in lasers, convection in a toroidal region, a disc dynamo, and a chaotic water wheel (Sparrow 4).

2. Chaotic Behavior

This section outlines the properties of the Lorenz System that are the fundamental reason it is now referred to as "chaotic." The section closes with an example taken from Lorenz's article that shows an example of a nonperiodic sequence within the Lorenz System.

2.1 Nonperiodic

For certain parameter values, the Lorenz System displays some interesting properties. Observe the trajectory of a particle, projected onto the X-Z-plane as shown in Figure 1. (It should be noted that the intersections in the path are merely a result of the projection and do not actually occur in three dimensions.) It displays turbulent behavior. The precise definition of turbulence varies depending on the context and purpose of the analysis, but one common turbulent characteristic is that the path depicted does not approach a periodic limit or an equilibrium point. In the center of each of the two loops is an unstable equilibrium point, and the particle orbits one, then the other, jumping back and forth in a manner that appears random, though is actually deterministic. Furthermore, the general form of this trajectory is not dependent upon initial conditions or integration method. Here the parameter values are $b = 8/3$, $\sigma = 10$, and $r = 28$, and the initial point is (10, 0, 10). If this system is perturbed slightly, though the details will change, the general form will remain.

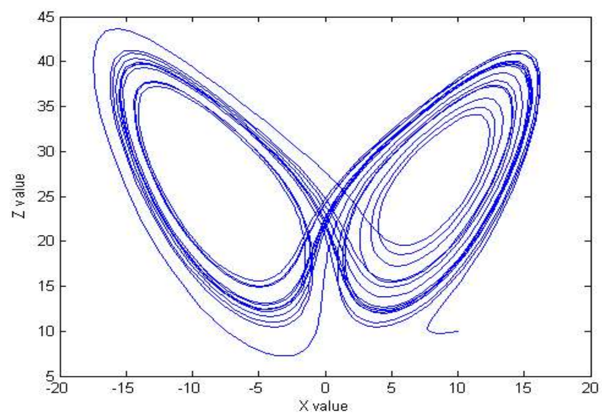

Figure 1 A numerical solution of the Lorenz equations projected onto the X-Z plane showing nonperiodic behavior.

Historically, closed physical systems were generally believed to approach periodic behavior. Nonperiodicity was an idea that was slow to come into acceptance. Lorenz acknowledges and thanks Saltzman for first pointing out the nonperiodic behavior of the convection equations (Lorenz 141).

2.2 Sensitive to Initial Conditions

One feature of the Lorenz System that had previously been largely ignored from analysis of physical systems is the fact that even a slight perturbation can alter the outcome drastically. This characteristic is also sometimes used in definitions of turbulence. The general line of thinking had previously been that a small difference in initial conditions would yield a small difference in results, except near unstable equilibria. As it turns out, this is not true for most real-world systems. Figure 2 illustrates this phenomenon in the Lorenz System. Two trajectories begin with very close initial conditions; in particular, $X_1 = X_2 = 10$, $Y_1 = Y_2 = 0$, $Z_1 = 10$, $Z_2 = 10.00000000001$. For the first 25 time units, the two trajectories seem identical. However, beyond 30 time units, they seem completely unrelated to each other.

It is this property of physical systems that makes long term prediction impossible, and ultimately dashes the hopes idealistic classical physicists. Strangely, this unpredictability comes from a completely deterministic system. Although Lorenz illustrated this phenomenon of sensitivity to initial conditions in a very simple and elegant manner in his paper, he was not the first to discover it. Poincaré once wrote, "[S]mall differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon" (Davies 53). Yet for a long time, people were reluctant to accept these ideas in favor of the non-chaotic, predictable universe.

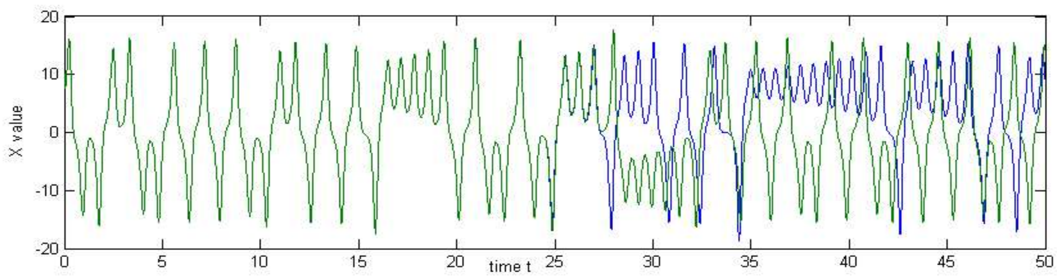

Figure 2 Two numerical solutions of the Lorenz equations showing sensitivity to initial conditions. The X values, plotted with respect to time, of two trajectories whose initial positions differ by one one-hundred-billionth in the Z direction. These two trajectories seem identical until around t = 27.5, at which point they completely separate forever.

2.3 Lorenz's Deterministic Nonperiodic Sequence

The following example comes from Lorenz's paper. It is used to illustrate some of the implications of the nonperiodic behavior of the sequence of *Z*-maxima. Lorenz wondered whether one *Z*-maximum could be used to predict the next.

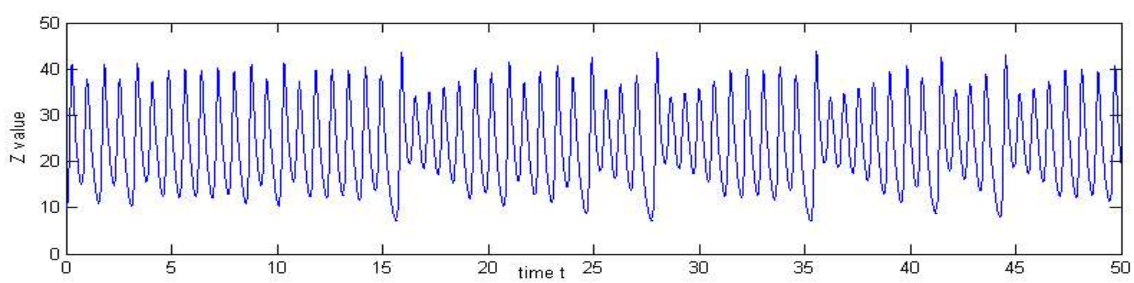

Figure 3 The Z value plotted against time to show the behavior of the sequence of Z-maxima.

It appears that once the *Z* value crosses a certain threshold, the particle will jump into an orbit around the other equilibrium, and *Z* will become suddenly small again. Again, the *Z* value will increase until it reaches a certain threshold, then jump back to the original orbit, and so on (Figure 3). He composed the following figure which plots M_n against M_{n+1}. That is, given one *Z*-maximum on the horizontal axis, the vertical axis shows the value of the next Z-maximum (Figure 4). The cusp corresponds to an orbit going to the origin, located on the local stable manifold.

Lorenz revealed the implications of this sequence using the following simplified example. Consider the sequence $M_{0,} M_{1,} \ldots$ of numbers between 0 and 1, defined as follows.

$$M_{n+1} = 2M_n \quad \text{if} \quad M_n < \frac{1}{2}$$

$$M_{n+1} \text{ is undefined if } M_n = \frac{1}{2}$$

$$M_{n+1} = 2 - 2M_n \quad \text{if} \quad M_n > \frac{1}{2}$$

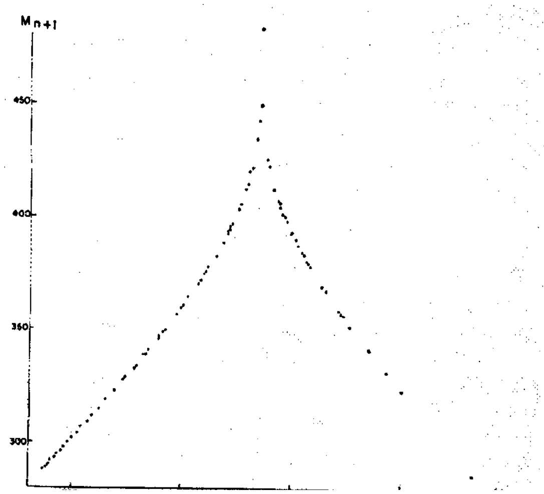

Figure 4 From Lorenz's paper: Corresponding values of relative maximum of and subsequent relative maximum of Z for the first 6000 iterations.

Thus, Lorenz's diagram is simplified to Figure 5. Given an initial M_0, the general form of M_n is given explicitly by

$$(*) \qquad M_n = m_n \pm 2^n M_0 \ ,$$

where m_n is an even integer. This can be shown inductively. Clearly $m_0 = 0$, and M_0 is of the desired form. Suppose the statement holds for $n = k$. There are two possibilities for the next iteration, $M_{k+1} = 2M_k = 2m_k \pm (2) 2^n M_0$ and $M_{k+1} = 2 - 2M_k = (2 - 2m_k) \mp (2) 2^k M_0$, both of which are of the desired form.

Now consider these cases.

Case 1: $M_0 = u/2^p$ where u is odd. Then by (*) and the fact that the sequence remains between 0 and 1, $M_{p-1} = 1/2$. Such sequences represent no convection at all, but there are only countably many of them.

Case 2: $M_0 = u/(2^p v)$ where u and v are relatively prime and odd. If $k > 0$ then $M_{p+1+k} = u_k / v$ where u_k is even and relatively prime to v. The number of possibilities for fractions $0 < \frac{u_k}{v} < 1$ is finite, so the sequence is periodic. Again, there are countably many such sequences.

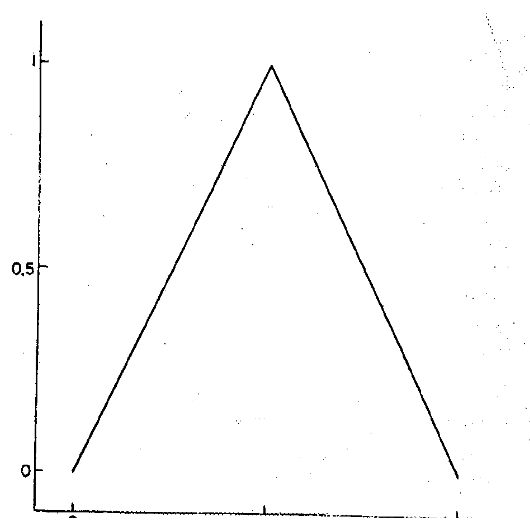

Figure 5 From Lorenz's paper: An idealization of Figure 4 using the M-sequence defined.

Case 3: M_0 is irrational. By (*) we have that for periodic sequences, $M_0 - M_k$ must be rational (since $M_k = \frac{m_k}{1 \mp 2^k}$ is rational). The sequence is thus not periodic, but it still could be quasi-periodic, approaching a periodic sequence asymptotically. To eliminate this possibility, Lorenz showed that all sequences are unstable to slight perturbations. Consider two sequences, $M_{0,} M_{1,} \ldots$ and $M_0', M_1', \ldots$ where $M_0' = M_0 + \epsilon$ for a small ϵ. Then for $k > 0$ we have $M_k' = M_k \pm 2^k \epsilon$, and the sequence is unstable. Therefore there are uncountably many nonperiodic sequences, corresponding to the irrational numbers between 0 and 1.

It is interesting to note that in this example, the existence of chaotic behavior corresponds directly to the existence of irrational numbers, a fact that people were once even more reluctant to accept. According to legend, when Hippasus first proved the existence of irrational numbers, he was thrown by the other Pythagoreans from a boat into the middle of the ocean[3].

3 Properties and Bifurcations

There exist far too many interesting features of the Lorenz System to discuss in a single paper. Henceforth the scope of this paper will be limited to some of the behavior of the system near the equilibria as *r* increases from an infinitesimal value toward infinity, for fixed values of the other parameters. To read about the system's other features, a good place to start is Sparrow's book (see references).

3.1 Dissipative

A system is dissipative if every orbit eventually moves away from infinity. That is, $\exists B \subset \mathbb{R}^2$ bounded, such that $\forall x^0 \in \mathbb{R}^2, \exists t_0$ (depending on x^0, B) with the solution $\phi(t, x^0)$ satisfying $\phi(t, x^0) \in B \, \forall t \geq t_0$ (Hale & Koçak 394). It can be shown that the Lorenz System is dissipative by using the Liapunov function

$$V = rX^2 + \sigma Y^2 + \sigma (Z - 2r)^2 \quad \text{(Sparrow 196).}$$

Then

$$\begin{aligned} \dot{V} &= 2rX\dot{X} + 2\sigma Y \dot{Y} + 2\sigma (Z - 2r)\dot{Z} \\ &= -2\sigma (rX^2 + Y^2 + bZ^2 - 2brZ) \ . \end{aligned}$$

Choose the bounded region *D* such that $\underline{X} \in D \Leftrightarrow \dot{V}(\underline{X}) \geq 0$, and let *c* be the maximum of *V* in *D*. Let *E* be the ellipsoid defined by $V \leq c + \epsilon$ for small $\epsilon > 0$. Then

$$\begin{aligned} &\underline{X} \notin E \Rightarrow \underline{X} \notin D \\ &\Rightarrow \dot{V}(\underline{X}) \leq -\delta \quad \text{for some } \delta > 0 \ , \end{aligned}$$

and the points on the trajectories passing through $\underline{X}$ will be associated with a decreasing V. Thus the trajectories will eventually enter and remain in E.

It follows from the fact that the divergence of the system is negative, $-(\sigma + b + 1)$, that the volume of this region will decrease with $e^{-(\sigma+b+1)t}$, so the set toward which all trajectories tend has zero volume (Sparrow 198).

3.2 Symmetric

The Lorenz System is invariant under the symmetry $(X, Y, Z) \rightarrow (-X, -Y, Z)$:

$$
\begin{aligned}
&-\dot{X} = \sigma(-(-X)+(-Y)) \\
&\qquad \Rightarrow \dot{X} = \sigma(-X+Y) \\
&-\dot{Y} = r(-X)-(-Y)-(-X)Z \\
&\qquad \Rightarrow \dot{Y} = rX - Y - XZ \\
&\dot{Z} = -b(Z)+(-X)(-Y) = -bZ + XY \quad .
\end{aligned}
$$

The invariance of the Z-axis implies that all trajectories on the Z-axis remain on the Z-axis and approach the origin. Furthermore, since

$$X=0, Y>0 \Rightarrow \dot{X}>0$$

and $$X=0, Y<0 \Rightarrow \dot{X}<0 \quad ,$$

all trajectories that rotate around the Z-axis must move clockwise with increasing time.

3.3 Equilibria

Let us now find the equilibrium points of the system. Solving

$$
\begin{aligned}
&\sigma(-\overline{X}+\overline{Y})=0 \\
&r\overline{X}-\overline{Y}-\overline{X}\,\overline{Z}=0 \\
&-b\overline{Z}+\overline{X}\,\overline{Y}=0
\end{aligned}
$$

yields

$$\overline{X}=0 \quad \text{, or} \quad \pm\sqrt{b(r-1)} \quad .$$

Depending upon the parameter values, we might have as equilibria

$$
\begin{aligned}
&(0,0,0) \quad , \\
&C_1=(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1) \quad \text{, and} \\
&C_2=(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1) \quad ,
\end{aligned}
$$

though the origin is always an equilibrium point.

The behavior of the Lorenz System is quite complex. To examine some simple bifurcations, first consider the case where two of the parameters are fixed at $b = 8/3$ and $\sigma = 10$, and let $0 < r < 1$. Then the above root has an imaginary part, and the only real equilibrium is

$$\overline{X}=\overline{Y}=\overline{Z}=0 \quad .$$

In fact, this equilibrium point is a global attractor for $0 < r < 1$. To see this, consider the Liapunov function,

$$V=X^2+\sigma Y^2+\sigma Z^2 \quad .$$

Then,

$$\begin{aligned}\dot{V} &= 2X\dot{X}+2\sigma Y\dot{Y}+2\sigma Z\dot{Z} \\ &= 2\sigma[(1+r)XY-X^2-Y^2-bZ^2] \\ &\Rightarrow \dot{V}<0\ \forall\ X,Y,Z \quad .\end{aligned}$$

This last inequality is seen by observing that

$$(1+r)XY-X^2-Y^2<0 \quad \text{, i.e.} \quad r<\frac{X^2+Y^2}{XY}-1 \quad ,$$

since if $X \geq Y$ (similar for the opposite inequality) then

$$\frac{X}{Y}-1\geq 0 \quad \text{, and} \quad 1-\frac{Y}{X}\geq 0 \quad \text{,so their product,}$$
$$\frac{X}{Y}-2+\frac{Y}{X}\geq 0 \qquad \Rightarrow \frac{X}{Y}+\frac{Y}{X}-1=\frac{X^2+Y^2}{XY}-1\geq 1>r \quad .$$

Thus beginning at any point away from the origin, the associated value of V must decrease, and the trajectory will approach the origin.

3.4 The First Bifurcations and "Preturbulence"

At $r = 1$ there is a bifurcation, and the other two equilibria appear. By the symmetry previously shown, we see that this is a pitchfork bifurcation. The origin becomes unstable, and two stable equilibria emerge, $C_{1,}C_2$, as seen in the bifurcation diagram in Figure 6.

Linearizing the system near an equilibrium point $(\overline{X},\overline{Y},\overline{Z})$ using the Jacobian matrix gives

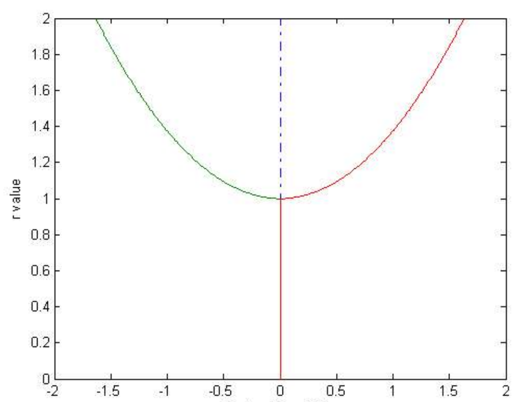

Figure 6 Bifurcation diagram varying r from 0 to 2 and fixing the other parameters, showing the X value of the equilibria.

$$\begin{bmatrix}\dot{X}\\ \dot{Y}\\ \dot{Z}\end{bmatrix}=\begin{bmatrix}-\sigma & \sigma & 0\\ r-\overline{Z} & -1 & -\overline{X}\\ \overline{Y} & \overline{X} & -b\end{bmatrix}\begin{bmatrix}X\\ Y\\ Z\end{bmatrix} \quad ,$$

and setting the determinant minus λI equal to zero gives the eigenvalues as solutions of

$$\lambda^3+(b+\sigma+1)\lambda^2+(b+b\sigma+\sigma-r\sigma+\sigma\overline{Z}+\overline{X}^2)\lambda+b\sigma(1-r)+\sigma(\overline{X}\,\overline{Y}+\overline{X}^2+b\overline{Z})=0 \quad .$$

If the equilibrium point is taken to be the origin, this simplifies to

$$\lambda^3+(b+\sigma+1)\lambda^2+(b+b\sigma+\sigma-r\sigma)\lambda+b\sigma(1-r)=0 \quad .$$

Since -*b* is clearly a solution, we factor to get

$$(\lambda+b)(\lambda^2+(\sigma+1)\lambda+\sigma(1-r))=0 \quad ,$$

and the three eigenvalues are:

$$\lambda_1,\lambda_2=\frac{-\sigma-1\pm\sqrt{(\sigma+1)^2+4\sigma(r-1)}}{2}$$
$$\lambda_3=-b \quad ,$$

expressed in such a way to make it clear that $\lambda_1>0$, $\lambda_2,\lambda_3<0$ for $r > 1$. Thus the origin becomes unstable (Hale & Koçak, Theorem 9.3). This is generally called a saddle, with a one-dimensional, unstable manifold.

The eigenvalues of a linearization near C_1 and C_2 simplify to be the solutions of

$$\mu^3+\mu^2(\sigma+b+1)+\mu b(\sigma+r)+2\sigma b(r-1)=0 \quad .$$

If we let $b = 8/3$ and $\sigma = 10$, then all three roots will have negative real part if

$$r<\frac{\sigma(\sigma+b+3)}{\sigma-b-1}=\frac{470}{19}\equiv r_H \quad \text{(Sparrow 10).}$$

Thus if $r < r_H$ then C_1 and C_2 are stable. For $r > r_H$, the two complex eigenvalues have positive real part, and the equilibria become unstable (Hale & Koçak, Theorem 9.3). At $r = r_H$ there is a subcritical Hopf bifurcation. So for values of r above this critical value, there are three unstable equilibria, yet it has already been shown that no trajectories

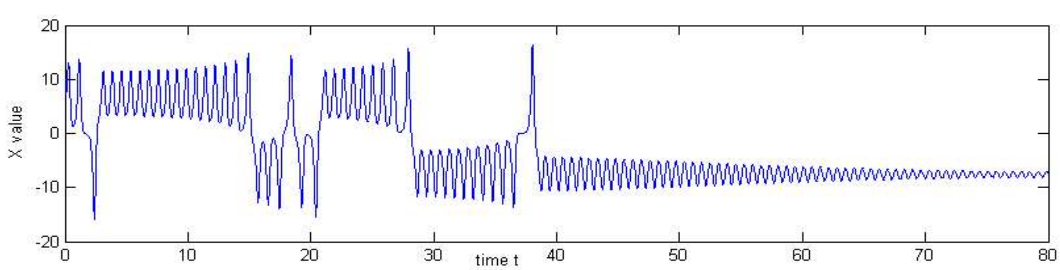

Figure 7 A "preturbulent" trajectory at r = 22.7, just before the bifurcation into chaotic behavior.

approach infinity but rather eventually enter a region around the origin. This is where we begin to see the chaotic behavior similar to that originally seen in Figure 1.

How does the system transition from nonchaotic to chaotic? To see intuitively what happens to the trajectories as r approaches this critical value, let us look at $r = 22.7$, just before the bifurcation (Figure 7). C_1 and C_2 are still stable, so the trajectory eventually spirals in toward one of them, but before it comes sufficiently close to an equilibrium, it exhibits "preturbulent" or "chaotic transcient" behavior.

3.5 Period Doubling Windows

For the values $99.524 < r < 100.795$ there exists what is called a "period doubling window." The first period doubling, listed in increasing order of period, occurs at $r \approx 99.98$. Just above this bifurcation value, trajectories approach a stable periodic orbit that circles the first equilibrium once, then the second equilibrium twice, which we will denote [1-2-2], as seen in Figure 8. As r decreases, the period doubles to [1-2-2-1-2-2] for $99.629 < r < 99.98$ (Figure 9). As r continues to approach the lower boundary of this window, 99.524, there is a cascade of period doubling similar to the behavior of chaotic one-dimensional maps. For $99.547 < r < 99.629$ there is a period of $[1\text{-}2\text{-}2]^4$, for $99.529 < r < 99.547$ there is a period of $[1\text{-}2\text{-}2]^8$, and so on (Figures 10 and 11 respectively).

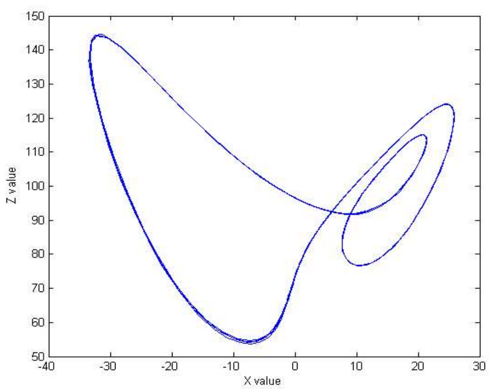

Figure 8 Stable periodic orbit for r = 100.5.

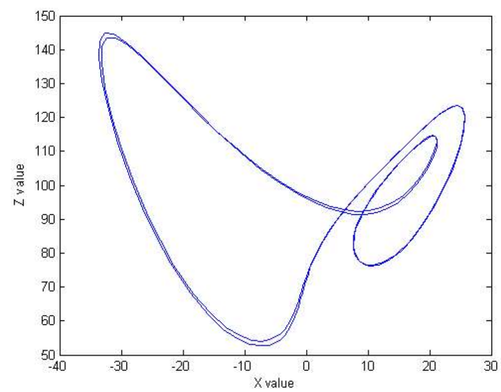

Figure 9 Stable periodic orbit after the first period doubling, for r = 99.7.

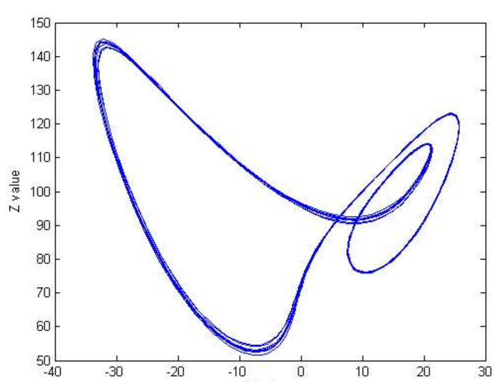

Figure 10 Stable periodic orbit after the second period doubling, for r = 99.6.

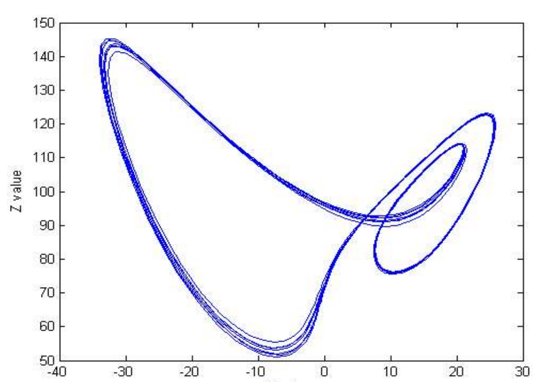

Figure 11 Stable periodic orbit after the third period doubling, for r = 99.537.

Not much is known of the particular behavior for values of r just less than the accumulation point of such cascades, where chaos begins. In general, the period doubling cascades have the same properties as in scalar maps, such as the Feigenbaum number, $\delta = \lim_{n\to\infty} \frac{r_{n+1}-r_n}{r_{n+2}-r_{n+1}} = 4.669\ldots$ which can be used to find the accumulation value r_∞.

The behavior in the upper half of the period doubling window is interesting as well. The fact that the window has an upper bound suggests that the system demonstrates nonperiodic behavior once again as r exceeds it. This is true, and the transition can be seen through what is called "intermittent chaos." This phenomenon is shown in Figure 12 for r = 100.93. As time progresses, the trajectory tends toward the periodic orbit, but every so often it lapses into nonperiodic, chaotic behavior for an interval of time. If r is within the period doubling window, the intermittent chaos will eventually cease, leaving a periodic orbit, but once r is beyond the upper bound, r_c, intermittent chaos will occur after any given time t'. As r moves further from the window, the periods of intermittent chaos will increase in length until they dominate the trajectory. In fact, the mean length of the periodic intervals seems to vary at a rate proportional to $(r-r_c)^{-1/2}$ (Sparrow 63)[4].

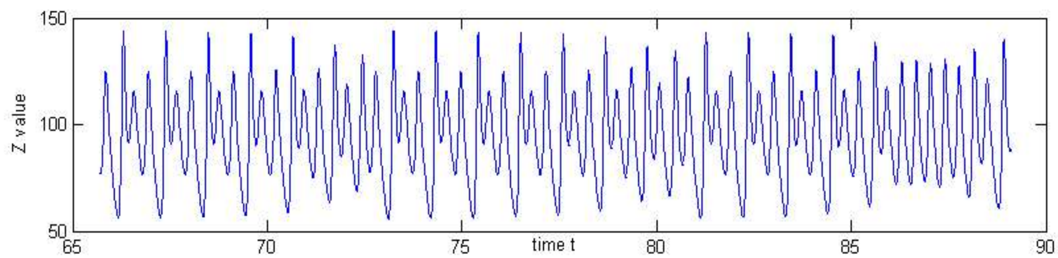

Figure 12 Intermittent chaos just above the period doubling window, for r = 100.93.

There exist two other period doubling windows as r increases. The first is 145 < r < 166. For 154.4 < r < 166.07 there is a stable symmetric (i.e. with the same symmetry as shown for the system itself) periodic orbit with a period described by [1-1-2-2]. At r ≈ 154.4 the stable symmetric orbit splits into two stable asymmetric periodic orbits with periods described by [1-1-2-2], producing between them an unstable periodic orbit. These orbits undergo simultaneous period doubling bifurcations as r decreases in a manner similar to that of the first window (Sparrow 59).

The final period doubling window is for 214.364 < r, with period described by a symmetric stable [1-2] orbit. This window is similar to the previous, except that for r > 313, the lowest period orbit continues to exist. There is no intermittent chaos as r increases. In fact, for large enough r, it is believed that this stable periodic orbit unioned with the three equilibria compose all of the non-wandering set, though this simple behavior depends on our particular choice of b and σ (for a theoretical justification of this claim, see Sparrow, chapter 7)[5].

4 Afterword

Lorenz's paper has spawned many deep and detailed analyses of this system. For a fairly rigorous discussion of other features of the system, such as homoclinic explosions, manifolds, another derivation, etc., see Sparrow's book. For an entertaining, qualitative dramatization of chaos and its implications, see Gleick's book. Lorenz's article itself is an invaluable milestone in physics and mathematics, and is highly recommended. Although it was published in the Journal of Atmospheric Sciences, it is essentially a mathematics article and is elegant in its simplicity.

Many higher dimensional systems have been designed as extensions of the Lorenz System. In fact, the system itself is a simplification of fluid convection, and higher order systems of fluid convection have been studied. Extensions of the Lorenz System have similar symmetries and demonstrate similar behavior, though more complex. In one such example, there are two equilibrium tori that act like the equilibria in the Lorenz System, with trajectories orbiting one torus, then the other, in a nonperiodic manner. Examination of these systems generally involve analyzing the way the system changes with a changing parameter, r, just as in the Lorenz System.

The conclusion of Lorenz's paper relates his theory to the atmosphere. Bounded finite dimensional systems must eventually come arbitrarily close to any previous state. We could then expect an analogue in the weather—i.e. a point in time when the atmosphere seems to be in a state identical to to a previously observed state. If the system is nonchaotic, then the weather will remain arbitrarily close to its past behavior, and weather forecasting will be a breeze. On the other hand, if an analogue occurs followed by new weather patterns, no forecasting scheme could be correct both times, and the system is unpredictable.

By now it is generally accepted that real physical systems contain this inherent unpredictable quality. Sensitivity to initial conditions, sometimes dubbed the "butterfly effect," is commonly described using an old folk poem in which a misplaced nail causes a kingdom to fall (see Gleick 23). Instead of repeating this example, I'd like to close with a Steinbeck passage that describes a real-life example of sensitivity to initial conditions that is, perhaps, a more accurate analogy.

> *"Two gallons is a great deal of wine, even for two paisanos. Spiritually the jugs may be graduated thus: Just below the shoulder of the first bottle, serious and concentrated conversation. Two inches farther down, sweetly sad memory. Three inches more, thoughts of old and satisfactory loves. An inch, thoughts of old and bitter loves. Bottom of the first jug, general and undirected sadness. Shoulder of the second jug, black, unholy despondency. Two fingers down, a song of death or longing. A thumb, every other song each one knows. The graduation stops here, for the trail splits and there is no certainty. From this point on anything can happen* (Steinbeck 43-4)*."*

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COMMENTS BY PROFESSOR ANDY FOSTER

[1] 'In fact, experimentalists would routinely <u>hide</u> (i.e. not publish) data from real systems exhibiting these types of behavior.'

[2] 'As we have seen, this is <u>usually</u> no problem, because of topological equivalence, etc. But ... the prevalent mindset is still that systems are classically predictable.'

[3] 'This "tent map" construction from the original flow is interesting, isn't it? Notice that the tent map is $\mathbb{R}^1$, so it is simpler than the Poincaré map, which would be $\mathbb{R}^2$.'

[4] 'For intermittent chaos, see Y. Pomeau and P. Manneville, *Comm. Math. Phys.* <u>74</u>, 189-197 (1980).'

[5] '...Did you know that the Lorenz system is <u>not</u> structurally stable? Yes, it has dense sets with and without saddle-saddle connections which can switch back and forth under arbitrarily small perturbations. So, what appears to be a stable chaotic set is <u>not</u> stable!'