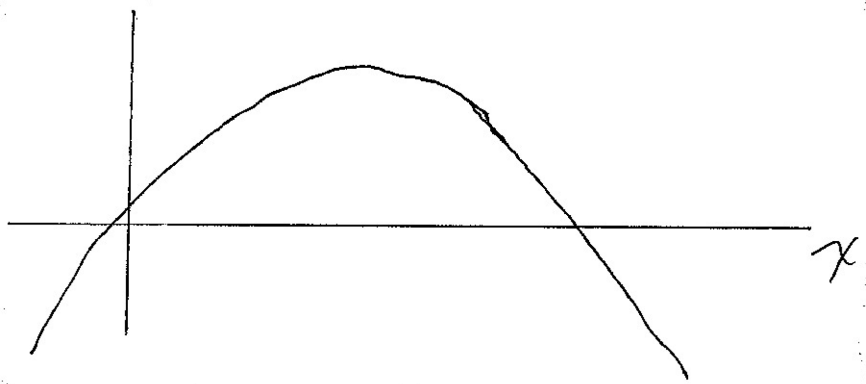
Logistic Map

The logistic map is characterized by the map function

$$M(x) = Rx(1 - x)$$

where x is the map variable and R a constant. This function is a simple parabola. The map is defined on the real axis as

$$x_{n+1} = M(x_n)$$



Origin: Insect population

The logistic map was first proposed as a model to describe insect population dynamics. Let z_n be the insect population at the n_{th} iteration. The iterations of the map $z_{n+1} = M(z_n)$ correspond to the time evolution of the insect population over generations. For small insect populations, when there is ample food, the population increases as $z_{n+1} = Rz_n$ per year. The population increases to $z_m = R^m z_0$ over m generations, which implies an exponential growth. But this trend must reverse itself due to the limit on the food supply. Underfed insects may begin to die before reaching maturity. The number of eggs laid per hatched insect will become less than that implied by Rz_n. The simplest model to incorporate this control over the insect population is to assume that the number of eggs laid per insect decreases linearly with the insect population, $Rz_n(1 - z_n/Z)$, where Z is the critical insect population beyond which the food supply is so scarce that no insect can lay eggs. This yields to

$$z_{n+1} = Rz_n(1 - z_n/Z)$$

or, dividing by Z,

$$z_{n+1} = Rz_n(1 - z_n)$$

Properties of the Logistics Map

The logistic map function

$$M(x) = Rx(1 - x)$$

is a simple parabola.

The origin, $x = 0$, is a fixed point of the map, i.e., $M(0) = 0$. The parabola has a second crossing at $x = 1$, i.e., $M(1) = 0$. The maximum of $M(x)$ is at $x = 1/2$, with $M(1/2) = R/4$. It is clear from the graph that any trajectory starting with negative x values will lead to an iterated x escaping to $-\infty$. Likewise, trajectories starting with positive $x > 1$ values will lead to an iterated x escaping to $+\infty$. From another point of view, the derivative of the map is

$$M'(x) = R(1 - 2x)$$

Therefore the origin, $x = 0$, is an unstable fixed point for all positive R.

The trajectories starting with x in the interval $0 \leq x \leq 1$ will only lead to $0 \leq x \leq 1$ for any $0 \leq R \leq 4$. Under these conditions, the logistic map exists on the real axis in the domain $0 \leq x \leq 1$.

Periodic Orbits

The behavior of the logistic map as a function of R is interesting, but very complex. This is illustrated by the search for periodic trajectories over a range in R located values around $R = 3$.

A periodic orbit is a trajectory that retraces itself some number of times given by the periodicity. In the one dimensional logistic map this implies that one or more points on the real axis will be visited over and over through the iterations. For instance, a period 1 orbit is found by requiring that a value of x be revisited after one iteration of the map, namely

$$M(x) = x$$

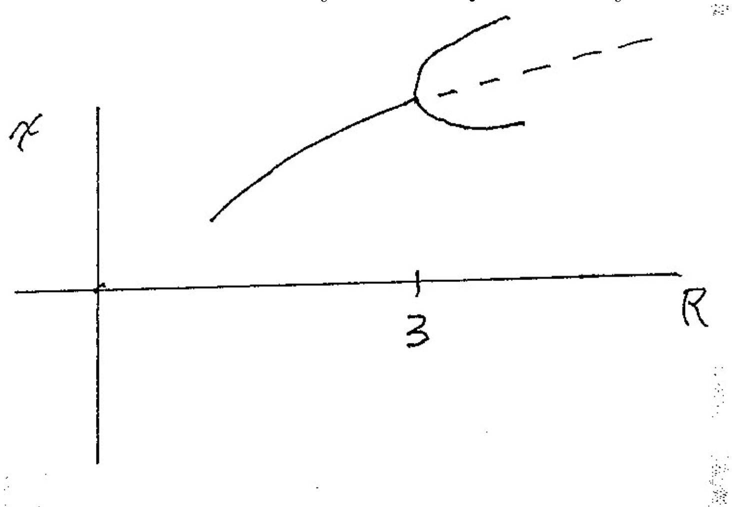
This yields an equation, $x = Rx(1 - x)$ which has 2 solutions, $x = 0$ and $x = 1 - 1/R$. We already know that the fixed points at the origin is unstable. The derivative of the map at the second fixed point takes the value $M'(1 - 1/R) = 2 - R$. Therefore this fixed point is stable if $|2 - R| \leq 1$, or $1 \leq R \leq 3$. For R within this range of values this stable fixed point is an attractor, i.e., trajectories starting from different initial values of x will migrate toward this P1 orbit. It can be shown that no other periodic orbits exist in this interval in R.

For $R > 3$, the P1 orbit at $x = (1 - 1/R)$ becomes unstable. Then, we may ask for other (higher) periodicity orbits. For instance

$$M^2(x) = M(M(x)) = x$$

would characterize a period two, P2, orbit. The value of x that solves this equation, $x2$ will be visited twice per close orbit. This solution, $x2$ exists for $R > 3$. Note that the period one trajectory still exists at $x = x_2$ but is unstable.

In scanning for R from below 3, $R < 3$, to above 3, $R > 3$, the logistic map undergoes a *bifurcation* from a stable period one (P1) trajectory to an unstable period one (P1) orbit and a new stable period two (P2) orbit. This is what is referred to as a *Period Doubling Bifurcation* in non-linear dynamics. Qualitatively



(from *Chaos in Dynamical Systems*, Edward Ott)