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### Electricity and Magnetism

We previously hinted a links between electricity and magnetism, finding that one can induce electric fields by changing the flux of a magnetic field through a wire, and finding that the two constants for electricity and magnetism were related to the speed of light. In this lecture, we follow through to discover the complete unit of the two forces, and the general physics of oscillations.

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# 1   Maxwell's Equations

Ampère's law, which states that

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I$$

does not properly account for what happens near a capacitor. JC Maxwell proposed adding a displacement current,

$$I_d = \epsilon_0 \frac{d\Phi_E}{dt}$$

where $\epsilon_0$ is the permittivity of free space given as $\epsilon_0 = 8.85418782 \times 10^{-12} m^{-3} kg^{-1} s^4 A^2$. The new version of the law becomes:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 (I + I_d)$$

To distinguish the *displacement* current from a normal one, we call normal currents *conduction currents*. Where did this equation come from? Recall that $V = Ed$, $Q = CV$ and $C = \epsilon_0 A/d$. We can write the displacement current as

$$I_D = \frac{\Delta Q}{\Delta t} = \frac{\Delta CV}{\Delta t} = \frac{\epsilon_0 \Delta (A/d) Ed}{\Delta t} = \epsilon_0 \frac{AE}{t} = \epsilon_0 \frac{\Delta \Phi_E}{\Delta t}$$

1. Show that the displacement current gives the necessary result.

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**Solution:** Where we previously found that the electric field created by an ideal conductor is $E = q/(\epsilon_0 A)$, so that

$$\Phi_E = EA = \frac{q}{\epsilon_0}$$

so that the contribution due to the displacement current is

$$\mu_0 \epsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \frac{dq}{dt}$$

which is exactly the same as if a wire were passing through.

What does it mean? It means that Ampère's law must also take into account the changing flux of the electric field–the magnetic field has now been related to the changing flux of the electric field just as an induced electric field was related to the changing flux of the magnetic field.

In addition to this and many other contributions, Maxwell collected all of the laws for electromagnetism into a set of four laws which we call Maxwell's equations:

$$
\begin{aligned}
\text{Gauss's Law (E)} \quad \oint \vec{E} \cdot d\vec{A} &= \frac{q}{\epsilon_0} \\
\text{Gauss's Law (M)} \quad \oint \vec{B} \cdot d\vec{A} &= 0 \\
\text{Faraday's Law} \quad \oint \vec{E} \cdot d\vec{s} &= -\frac{d\Phi_B}{dt} \\
\text{Ampère-Maxwell Law} \quad \oint \vec{B} \cdot d\vec{s} &= \mu_0 I + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt}
\end{aligned}
$$

## 1.1   EM force

The total force due to electrical and magnetic force is called the Lorentz force law:

$$ \vec{F} = q\vec{E} + q\vec{v} \times \vec{B} $$

## 1.2   A strange equation

The implications of the last of the two of Maxwell's equations are tremendous. In free space with no current, they take on very similar forms:

$$ \oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} $$

$$ \oint \vec{B} \cdot d\vec{s} = \epsilon_0 \mu_0 \frac{d\Phi_E}{dt} $$

Via a series of manipulations, these two equations can be recast as

$$ \frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} $$

$$ \frac{\partial B}{\partial x} = -\mu_0 \epsilon_0 \frac{\partial E}{\partial t} $$

Then

$$
\begin{aligned}
\frac{\partial^2 B}{\partial x^2} &= -\mu_0 \epsilon_0 \frac{\partial}{\partial x}\left(\frac{\partial E}{\partial t}\right) \\
&= -\mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(\frac{\partial E}{\partial x}\right) \\
&= \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(\frac{\partial B}{\partial t}\right) \\
&= \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}
\end{aligned}
$$

A similar line of reasoning produces the equation

$$
\frac{\partial^2 E}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}
$$

To uncover the true implications of these two equations, we must take a detour into chapter 15 and 16.

## 2   Wave Equations

Consider a pulse traveling along a string. The position of every point along that string can be written $y(x,t) = f(x+vt)$ for a pulse traveling to the left and $y(x,t) = f(x-vt)$ for a pulse traveling to the right. To help see why this is so, think about the pulse in time and on one specific place on the string, say $x = 0$. Then at $t = 0$, $y(0) = f(0)$. If the pulse is traveling to the right, then its position at $t = 0$, which is $f(0)$ will move over a distance $vt$ after $t$ seconds. That is, at position $x = vt$, we require $f$ to be equal to $f(0)$, which can be satisfied with the equation $f(x-vt) = 0$.

Now let's do some calculus on this equation. Let $u = x - vt$ and use the chain rule to find that

$$
\frac{\partial^2 y}{\partial x^2} = \left(\frac{du}{dx}\right)^2 \frac{\partial^2 f(u)}{\partial u^2} = \frac{\partial^2 f(u)}{\partial u^2}
$$

$$
\frac{\partial^2 y}{\partial t^2} = \left(\frac{du}{dt}\right)^2 = v^2 \frac{\partial^2 f(u)}{\partial u^2}
$$

Therefore we get,

$$
\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}
$$

So we see that Maxwell's equations lead to the wave equation, which implies that the velocity of the electromagnetic waves that Maxwell predicts is related to the permeability and permittivity of free space:

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Let's calculate those values:

$$v = \frac{1}{\sqrt{(8.85 \times 10^{-12} C^2/Nm^2)(4\pi \times 10^{-7} Ns^2/C^2)}} = 3.0 \times 10^8 m/s$$

This is precisely the speed of light. Maxwell's prediction was radical, and was confirmed when in 1887 (eight years after Maxwell's death) Hertz experimentally proved this with a spark-gap experiment.

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**ElectroMagnetic Waves and Light**

Electromagnetic waves propagate in free space at the speed of light; in fact, light is an electromagnetic wave and travels at a speed $c = 3.0 \times 10^8 m/s$. The actual number is $c = 2.99792458 \times 10^8 m/s$, but we round it without much harm for our goals.

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## 3　Nature of EM waves

Suppose we have a capacitor wired up to an AC source:



(a)
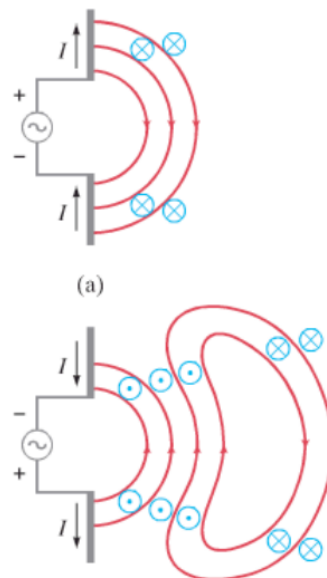
Since the source is sinusoidal, so too will the wave be sinusoidal in nature. Maxwell's equations tell us that a changing magnetic field produces a changing electric field and a changing electric field produces a changing magnetic field. In this sense, as the wave

propagates, it will be self sustaining, it will go on oscillating between electric and magnetic fields forever; though the intensity of the field will decrease the further away it travels from the source just as the brightness of the sun is stronger on Earth than it is on pluto.

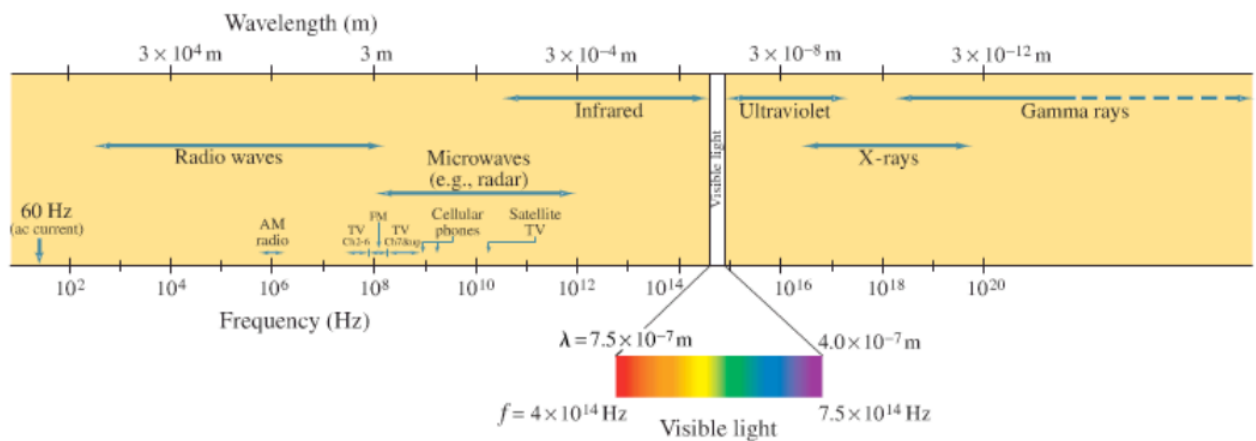These waves will propagate in a regular way, oscillating as the source oscillates:
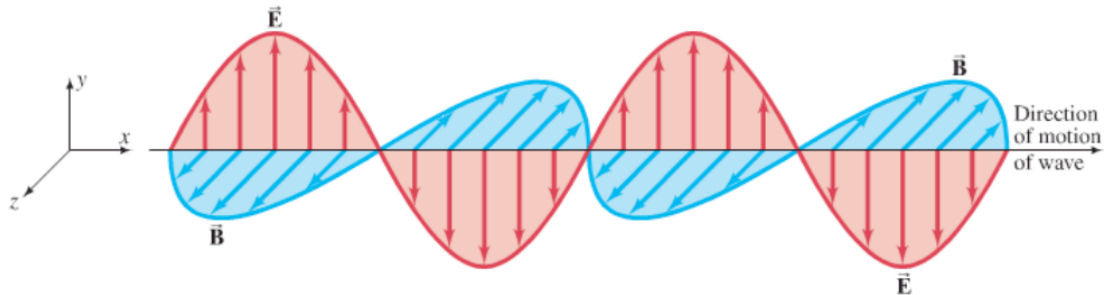


In this way, we can produce waves of varying frequency. The condition for a wave that relates its frequency to its wavelength is that,

$$v = \lambda f$$

or in the case of EM waves,

$$c = \lambda f$$

This results in the fact that there is an infinite spectrum of frequency/wavelength combinations of EM waves:



2. Calculate the wavelength of a 60 Hz EM wave; a 93.3 MHZ FM radio wave, and a beam of red light from a laser at frequency $4.74 \times 10^{14}$ Hz.

**Solution:** Since $c = \lambda f$ then

$$\lambda = \frac{c}{f} = \frac{3.0 \times 10^8 m/s}{60 s^{-1}} = 5.0 \times 10^6 m$$

which is approximately the radius of the Earth, and,

$$\lambda = \frac{c}{f} = \frac{3.0 \times 10^8 m/s}{93.3 \times 10^6 s^{-1}} = 3.22 m$$

which is approximately the height of two people stacked together, and

$$\lambda = \frac{c}{f} = \frac{3.0 \times 10^8 m/s}{4.74 \times 10^{14} s^{-1}} = 6.33 \times 10^{-7} m$$

, which is about a hundredth of the width of a human hair.

# 4   Power in EM waves

If AC current is driving the generation of waves, the most simple form of the wave can waves can be (in one-d version):

$$
\begin{aligned}
E &= E_{max} \cos(kx - \omega t) \\
B &= B_{max} \cos(kx - \omega t)
\end{aligned}
$$

where $k = 2\pi/\lambda$ and $\omega = 2\pi f$.

Now using the equation,

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$$

and,

$$\frac{\partial E}{\partial x} = -k E_{max} \sin(kx - \omega t)$$

$$\frac{\partial B}{\partial t} = \omega B_{max} \sin(kx - \omega t)$$

we find that

$$k E_{max} = \omega B_{max} \quad \rightarrow \quad \frac{E_{max}}{B_{max}} = \frac{\omega}{k} = c$$

We now introduce the Poynting vector which is the rate of transfer of energy by an electromagnetic wave,

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

for the type of waves we deal with in this chapter (called plane waves), this is simply

$$S = \frac{EB}{\mu_0}$$

Using $B = E/c$, then,

$$S = \frac{E^2}{\mu_0 c} = \frac{cB^2}{\mu_0}$$

which means we can express the energy transferred from an EM wave as a function of either the magnetic and electrical field of the wave.

Because the fields oscillate as a function of cos, and the power transferred is a function of the field squared, the energy transfer will be a function of cosine squared, and the average of cosine squared is $1/2$ so that the average value of energy of the wave is the intensity given as,

$$I = S_{avg} = \frac{E_{max}^2}{2\mu_0 c} = \frac{cB_{max}^2}{2\mu_0}$$

3. Radiation from the Sun reaches the Earth (above the atmosphere) at a rate of about 1350 J/sm$^2$. Assume that this is a single EM wave, and calculate the maximum values of $E$ and $B$.

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**Solution:** Given the average rate is

$$I = 1350 J/sm^2 = \frac{E_{max}^2}{2\mu_0 c}$$

we can solve for $E_{max}$,

$$E_{max} = \sqrt{2\mu_0 c 1350 J/sm^2} = 1.01 \times 10^3 V/m$$

$$B_{max} = \frac{E_{max}}{c} = \frac{1.01 \times 10^3 V/m}{3.00 \times 10^8 m/s} = 3.37 \times 10^{-6} T$$

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