Name: ____________________________

> You may answer the questions in the space provided here, or if you prefer, on your own notebook paper.

Short answers

1. Why is c the ultimate speed limit in the universe?

> **Solution:**
>
> 1. Acceleration per given force decreases with speed.

2. The twin paradox is resolved by noticing that
   - A. The twins are actually the same way at the end of the trip.
   - B. The twin on the rocket is not in a preferred frame.
   - C. The twin on Earth is not in the preferred frame.
   - **D. One twin accelerates which radically shifts their spacetime experience relative to the other twin.**

3. How do we know that the universe is expanding?

> **Solution:** Nearly all galaxies are redshifted indicating that they are spreading apart.

4. What was a major clue for Einstein that Newtonian gravity was flawed?

> **Solution:** Mercury's orbit was inconsistent with predictions.

Short Problems

5. Two motorcycle pack leaders named David and Emily are racing at relativistic speeds along perpendicular paths. Emily is traveling south at $v_y = -0.90c$ and David is traveling East as $v_x = +0.75c$ as measured by a police officer standing still in the middle of an intersection.

   How fast does Emily recede as seen by David over his right shoulder?

Hint: It is often easier if you select one of the two frames of a problem to be one that is at rest relative to the problem (the cop) and the other frame to be the person whose perspective you are being asked to describe (David). In this case, Emily would be the third object whose velocity we wish to transform from the cop's frame to David's frame.

Solution: Recall that:

$$v'_x = \frac{v_x - u}{1 - \frac{uv_x}{c^2}}$$

$$v'_y = \frac{v_y}{\gamma\left(1 - \frac{v_x u}{c^2}\right)}$$

$$v'_z = \frac{v_z}{\gamma\left(1 - \frac{v_x u}{c^2}\right)}$$

Let's set horizontal as the $x-$axis and vertical as the $y-$axis. We identify that $v_x = u = 0.75c$ for David and $v_x = 0$ and $v_y = -0.90c$ for Emily as measured by the cop.

Transforming from the cop's frame to David's:

$$v'_x = \frac{v_x - u}{1 - \frac{v_x u}{c^2}} = \frac{0 - 0.75c}{1 - \frac{0.75c}{c^2}} = -0.75c$$

$$v'_y = \frac{v_y}{\gamma\left(1 - \frac{v_x u}{c^2}\right)} = -0.60c$$

So that finally,

$$v' = \sqrt{(v'_x)^2 + *v'_y)^2} = 0.96c$$

6. An electron ($m = 9.11 \times 10^{-31} kg$) moves with a speed of $0.1c, 0.5c$, and $0.9c$. In each case, find the percent difference between the relativistic momentum calculation and the classical momentum calculation.

Solution: The classical definition of momentum was $p_c = mv$. The relativistic definition is $p_r = \gamma mv$. The percentage difference is:

$$\delta p = 100 \times \frac{p_r - p_c}{p_c} = \frac{p_r}{p_c} - \frac{p_c}{p_c} = 100\left(\gamma - 1\right)$$

For $v = 0.1c$,

$$\delta p = 0.50\%$$

$$p_c = \left(9.11 \times 10^{-31} \text{ kg }\right)(0.1)\left(\left(2.99 \times 10^{8}\right)\frac{\text{m}}{\text{s}}\right) = 2.72 \times 10^{-23} kgm/s$$

$$p_r = \gamma\left(9.11 \times 10^{-31} \text{ kg }\right)(0.1)\left(\left(2.99 \times 10^{8}\right)\frac{\text{m}}{\text{s}}\right) = 2.74 \times 10^{-23} kgm/s$$

For $v = 0.5c$

$$\delta p = 15.47\%$$

$$p_c = \left(9.11 \times 10^{-31} \text{ kg }\right)(0.5)\left(\left(2.99 \times 10^{8}\right)\frac{\text{m}}{\text{s}}\right) = 1.36 \times 10^{-22} kgm/s$$

$$p_r = \gamma\left(9.11 \times 10^{-31} \text{ kg }\right)(0.5)\left(\left(2.99 \times 10^{8}\right)\frac{\text{m}}{\text{s}}\right) = 1.57 \times 10^{-22} kgm/s$$

For $v = 0.9c$,

$$\delta p = 129.42\%$$

$$p_c = \left(9.11 \times 10^{-31} \text{ kg }\right)(0.9)\left(\left(2.99 \times 10^{8}\right)\frac{\text{m}}{\text{s}}\right) = 2.45 \times 10^{-22} kgm/s$$

$$p_r = \gamma\left(9.11 \times 10^{-31} \text{ kg }\right)(0.9)\left(\left(2.99 \times 10^{8}\right)\frac{\text{m}}{\text{s}}\right) = 5.62 \times 10^{-22} kgm/s$$

7. The ^{216}Po nucleus is unstable and exhibits radioactivity. It decays to ^{212}Pb by emitting an alpha particle, which is a helium nucleus, ^{4}He. The relevant masses are $m_i = 216.001915u$ for ^{216}Po, $m_f = m(^{212}Pb) + m(^{4}He) = 211.991898u + 4.002603u$. $1u = 1.66x10^{-27}kg$.

   (a) Find the mass change in this decay.

   Solution:

   $$\Delta m = m_f - m_i = 211.991898u + 4.002603u - 216.001915u = -0.007414u$$

   In Kilograms:

   $$\Delta m = -1.230\,724 \times 10^{-29} kg$$

   (b) Find the resulting release of energy.

Solution: The negative sign indicates a release in energy, and be mindful that we need to use the kilogram number since we are seeking energy in Joules:

$$\Delta E = \Delta mc^2 = -1.100\,280 \times 10^{-12} J$$

8. The net nuclear fusion reaction inside the Sun can be written as $4^1H \rightarrow^4 He + E$. The rest energy of each hydrogen atom is $938.78MeV$, and the rest energy of the helium-4 atom is $3728.4MeV$.

   (a) Calculate the percentage of the starting mass that is transformed to other forms of energy.

   Solution: The percentage is given as:

$$\begin{aligned} Perc &= \frac{m_f - m_i}{m_i} \times 100\% && (0.1) \\ &= 100\% \times \frac{3728.4MeV - 4 \times 938.78MeV}{4 \times 938.78MeV} && (0.2) \\ &= -0.71\% && (0.3) \end{aligned}$$

   (b) If that energy were converted to the kinetic energy of a 1kg cannon ball, how fast would the cannon ball be traveling in meters per second?

   Solution:

$$\Delta E = \Delta mc^2 = (3728.4MeV - 4 \times 938.78MeV) \left(\frac{1 \times 10^6 eV}{MeV}\right) \left(\frac{1.602 \times 10^{-19}\text{ J}}{eV}\right)$$

   Solving,

$$\Delta E = -7.64 \times 10^{-11} J$$

   Assuming that is converted to KE, and using classic KE since we can already tell the speed will be small,

$$\Delta KE = -\Delta E = \frac{1}{2}mv^2 \rightarrow v = \sqrt{-2\Delta E/m}$$

$$v = 1.24 \times 10^{-5}\text{m/s}$$

   While that may not be an impressive speed, recall that this was from just 4 hydrogen atoms, and in fact, there are approximately 8.92×10^{56} hydrogen atoms in the sun.

Longer problems

9. A global positioning system (GPS) satellite moves in a circular orbit with period 11 h 58 min.

   (a) Determine the radius of its orbit. Recall from Physics I that for a circular orbit, the centripetal force and the gravitational force must balance so that

$$\frac{G \cdot M_E \cdot m}{r^2} = \frac{mv^2}{r}$$

   where $G = 6.67 \times 10^{-11} Nm^2/kg^2$ is the original gravitational constant and the mass of the Earth is $M_E = 5.98 \times 10^{24} kg$. We also recall that $v = 2\pi r/T$ and substitution yields:

$$r = \left(\frac{GM_E T^2}{4\pi^2}\right)^{\frac{1}{3}}$$

Solution: First convert the time to seconds:

$$11h58min = 43080s = T$$

$$r = \left(\frac{\left(6.6742 \times 10^{-11}\,\frac{\mathrm{m}^3}{\mathrm{kg\ s}^2}\right) 5.98 \times 10^{24}\ \mathrm{kg}\ \left(43080\mathrm{s}\right)^2}{4\pi^2}\right)^{\frac{1}{3}}$$

$$r = 2.66 \times 10^7\ \mathrm{m}$$

   (b) Determine its speed. Recall that $v = 2\pi r/T$.

Solution:

$$v = \frac{2\pi r}{T} = 3.88 \times 10^3\,\frac{\mathrm{m}}{\mathrm{s}}$$

Which gives

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = 1.000\,000\,000\,1\,\frac{\mathrm{m}}{\mathrm{s}}$$

   (c) The nonmilitary GPS signal is broadcast at a frequency of $1575.42 MHz$ in the reference frame of the satellite. When it is received on the Earth's surface by a GPS receiver, what is the fractional change in this frequency due to time dilation as described by special relativity? The fractional change is $\Delta f/f$, i.e. the change in frequency divided by the original frequency.

Hint: Our derivation for relativistic Doppler effect in class was only for when the receiver and/or source were moving along a straight line. In this case, the satellite is moving perpendicular to a receiver on Earth (circular motion) so we can instead use the time dilation equation equation. If the period of oscillation is ΔT on the satellite, then the period would be measured $\Delta T/\gamma$ here on Earth. Using $f = 1/\Delta T$ and $f' = 1/(\Delta T/\gamma)$, find $\Delta f/f = \frac{f-f'}{f}$.

Solution:

$$\begin{aligned} f' &= \frac{1}{\Delta T\gamma} && (0.4) \\ &= \frac{\sqrt{1-\frac{v^2}{c^2}}}{\Delta T} && (0.5) \\ &= f\sqrt{1-\frac{v^2}{c^2}} && (0.6) \\ &= 6.347\,514\times 10^{-10}\frac{1}{\mathrm{s}} && (0.7) \end{aligned}$$

Then we have,

$$\Delta f/f = \frac{f-f'}{f} = 8.400\,345\times 10^{-11}$$

(d) Calculate this fractional change in frequency due to the change in position of the satellite from the Earth's surface to its orbital position.

Given:

$$U_g = -\frac{GM_E m_s}{r}$$
$$\Delta U_g = U_g(r_s) - U_g(r_E)$$

and the radius of the Earth is $r_E = 6.37\times 10^6 m$. You **will not** need to know the mass of the satellite, leave it as a variable m_s and it will cancel out eventually.

Solution:

First calculate ΔU_g:

$$\Delta U_g = -(GM_e m_s)\left(\frac{1}{r_s} - \frac{1}{r_E}\right) = m_s 4.763\,576\times 10^7 \text{ units}$$

Now recall that

$$\frac{\Delta f}{f} = \frac{\Delta U_g}{m_s c^2}$$

So that we have

$$\frac{\Delta f}{f} = \frac{m_s 4.763\,576 \times 10^7}{m_s c^2} = 5.328\,325 \times 10^{-10}$$

(e) What is the overall fractional change in frequency due to both time dilation and gravitational blueshift (add them up)? Which effect was bigger?

Solution:
GR had the bigger effect, and the total is the sum of the two effects:

$$\frac{\Delta f}{f} = 4.488\,291 \times 10^{-10}$$

Some possibly useful equations.

$$e = 1.602 \times 10^{-19}\ \mathrm{C}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

$$\Delta E_0^2 = \Delta m c^2$$

$$p = \frac{mu}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\mu_0 = \left(4 \times \pi \times 10^{-7} \mathrm{Tm/A}\right)$$

$$c = \left(2.99 \times 10^{8}\right) \frac{\mathrm{m}}{\mathrm{s}}$$

$$m_e c^2 = 0.511\ \mathrm{MeV}$$

$$K = (\gamma - 1)\, mc^2$$

$$K = E - mc^2$$

$$E^2 = p^2 c^2 + (m_p c^2)^2$$

$$x' = \gamma(x - vt)$$

$$t' = \gamma(t - \frac{vx}{c^2})$$

$$L' = \frac{L_p}{\gamma}$$

$$t' = \gamma t_p$$

$$\frac{\Delta f}{f} = \frac{\Delta U_g}{mc^2}$$

$$f = \sqrt{\frac{c+u}{c-u}} f_0$$

$$v'_x = \frac{v_x - u}{1 - \frac{u v_x}{c^2}}$$

$$v'_y = \frac{v_y}{\gamma\left(1 - \frac{v_x u}{c^2}\right)}$$

$$v'_z = \frac{v_z}{\gamma\left(1 - \frac{v_x u}{c^2}\right)}$$

$$K_{classic} = \frac{1}{2} m v^2$$

End of exam