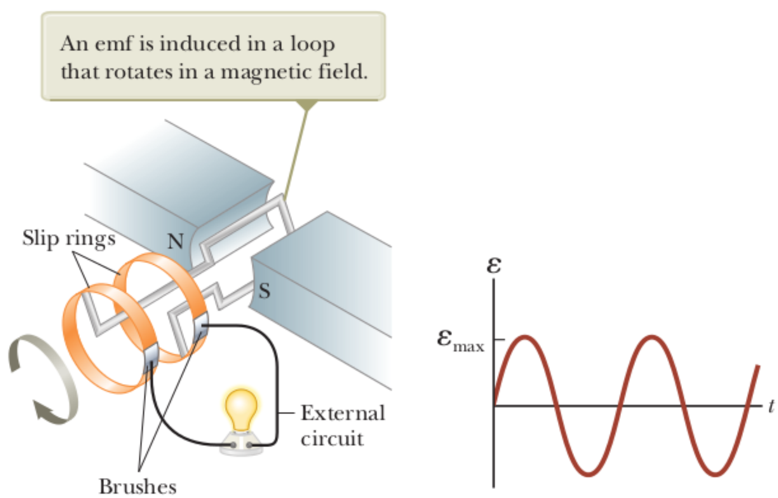
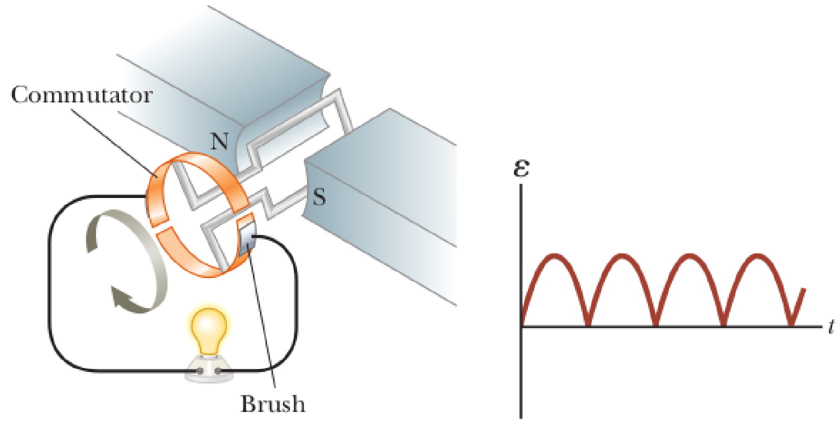
AC Circuits

Putting everything we've learned so far together, we make more complicated circuits with more interesting behavior.

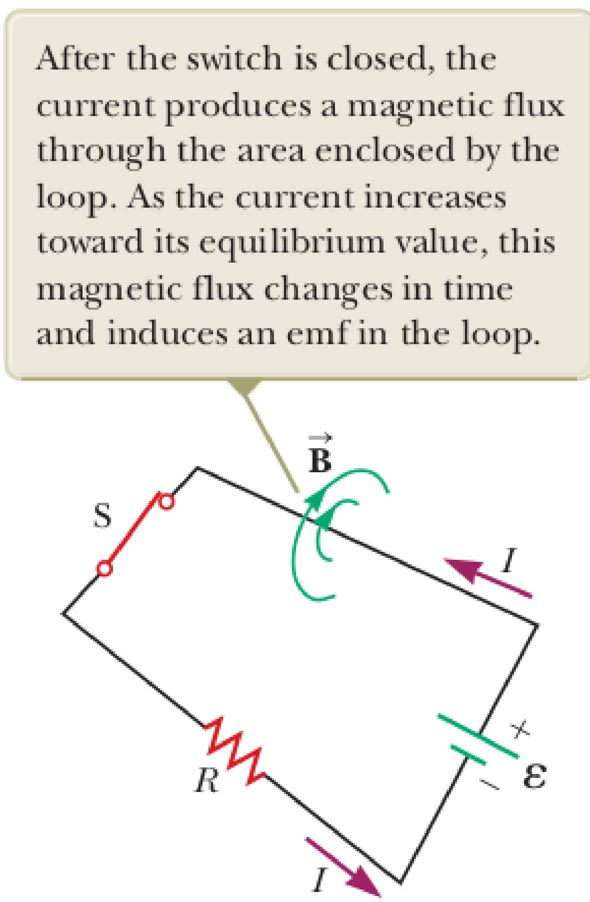
- **Faraday's Law**: $\mathcal{E} = -\frac{d\Phi_B}{dt}$, $\Phi_B = \oint \mathbf{B} \cdot d\mathbf{A}$.
- For a coil of N loops, $\mathcal{E} = -N\frac{d\Phi_B}{dt}$
- Motional emf is a voltage difference induced by moving a conductor through a constant magnetic field. Since $F_b = q\mathbf{v} \times \mathbf{B}$ or $F_b = I \times \mathbf{B}$, we see that positive and negative charges will experience an opposite force due to motion and thus self-segregate to either side of the conductor.
- **Lenz's law** helps us interpret this: *The induced current in a loop is in the direction that creates a magnetic field that opposes the change in magnetic flux through the area enclosed by the loop.*
- A changing magnetic field in free space induces an electric field in free space! You don't even need wire (**example**).
- **AC (alternating-current) generator.**:



- How can we rotate this loop to generate electrical power?
- The flux through a rotating loop of N turns is $\Phi_B = BA\cos(\omega t)$
- The induced emf is then: $\mathcal{E} = -N\frac{d\Phi_B}{dt} = NAB\omega\sin(\omega t)$.
- The max voltage is $\mathcal{E}_{max} = NAB\omega$.
- **Direct current generator:**



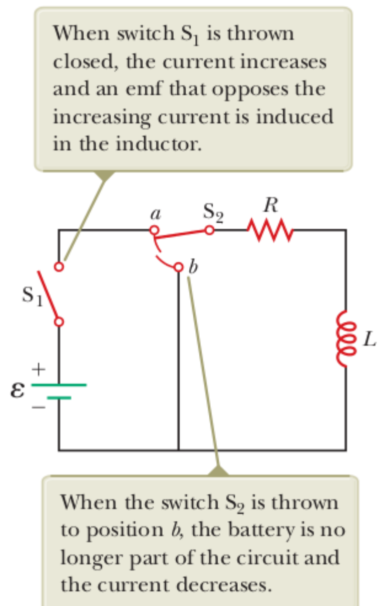
- Induced EMF



- Each circuit may have a different self-inductance proportional to the current passing through it, which we right as: $\mathcal{E} = -L\frac{dI}{dt}$.
- SI unit of inductance is called **henery** H, 1 H = 1 V·s/A.

- For solenoids, $\mathcal{E}_L = -N\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$, so $L = N\Phi_B/I$.
- Self-inductance of circuit can be represented as an inductor:

- **RL Circuits**:



which behaves with the equation

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0$$

with solution

$$I = \frac{\mathcal{E}}{R}\left(1 - e^{-Rt/L}\right) = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau}\right)$$

The value,

$$\tau = \frac{L}{R}$$

is called the RL circuit time constant.

- This is why real currents don't go from 0 to I instantly.
- We switch to b, and find that the current behaves as:

$$I = \frac{\mathcal{E}}{R}e^{-Rt/L} = \frac{\mathcal{E}}{R}e^{-t/\tau}$$

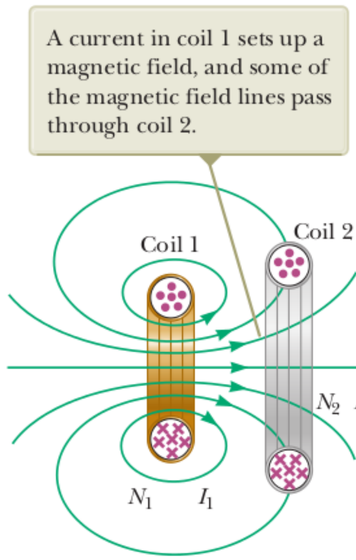
- Energy in magnetic field can be found by:

$$I\mathcal{E} = I^2R + LI\frac{dI}{dt}$$

so that

$$\frac{dU}{dt} = LI\frac{dI}{dt} \rightarrow U = \frac{1}{2}LI^2$$

- Mutual inductance:



Flux created by one coil induces current in a nearby coil. The two circuits will have a value called **Mutual Inductance**:

$$M_{12} = \frac{N_2\Phi_{12}}{I_1}$$

where circuit 1, with current I_1, sets up a magnetic flux Φ_{12} in the second coil with N_2 turns.

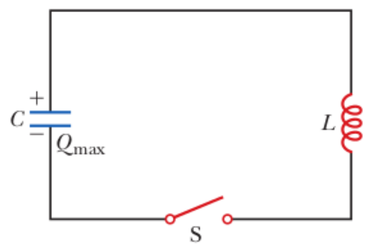
- The voltage in the second coil is,

$$\mathcal{E}_2 = -N_2\frac{d\Phi_{12}}{dt} = -N_2\frac{d}{dt}\left(\frac{M_{12}I_1}{N_2}\right) = -M_{12}\frac{dI_1}{dt}$$

- The current changing in the second coil, in turn, induces,

$$\mathcal{E}_1 = -M_{21}\frac{dI_2}{dt}$$

- **LC Circuit**:



Assuming ideal circumstances (no resistors, ignore radiation), the energy in the circuit is:

$$U = U_c + U_L = \frac{Q^2}{2C} + \frac{1}{2}LI^2$$

and

$$\frac{dU}{dt} = \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} \rightarrow \frac{d^2Q}{dt^2} = -\frac{1}{LC}Q$$

whose solution is,

$$Q = Q_{max}\cos(\omega t + \phi)$$

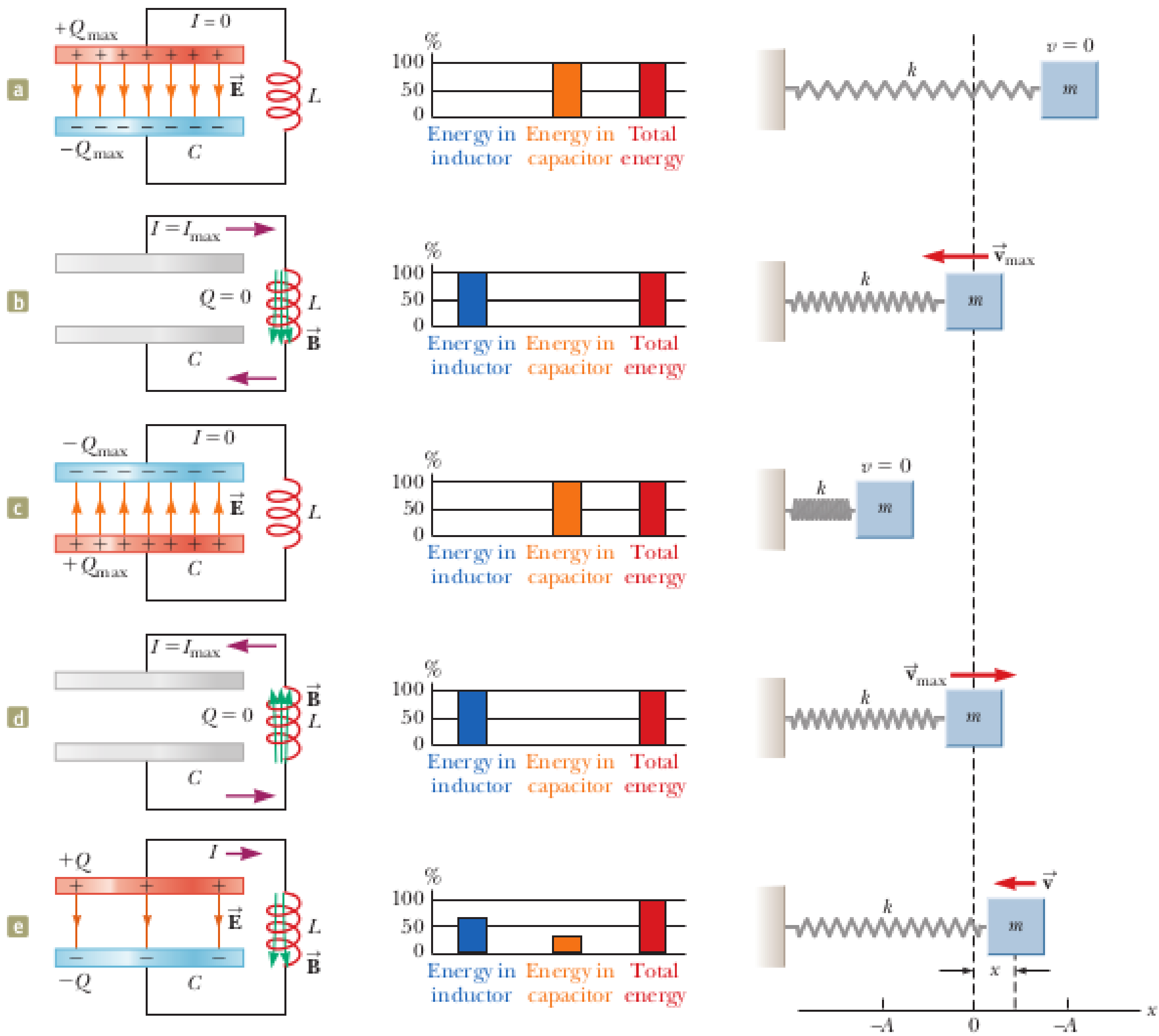
where $\omega = 1/\sqrt{LC}$ and ϕ is a phase constant that depends on the initial state of the circuit at time $t = 0$. If $I = 0$ at $t = 0$, then $\phi = 0$. In such a case,

$$Q = Q_{max}\cos(\omega t)$$

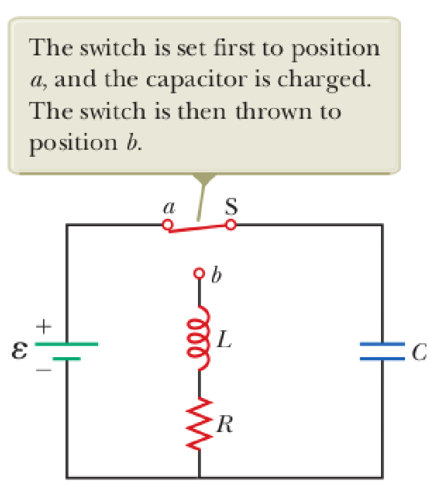
and

$$I = -\omega Q_{max}\sin(\omega t)$$

In other words, the circuit cycles back and forth forever!



- What happens if we throw a resistor in the circuit?



The change in energy due to a resistor is,

$$\frac{dU}{dt} = -I^2 R$$

which sets up the equation:

$$LI\frac{dI}{dt} + \frac{Q}{C}\frac{dQ}{dt} = -I^2 R$$

which we can recast as,

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

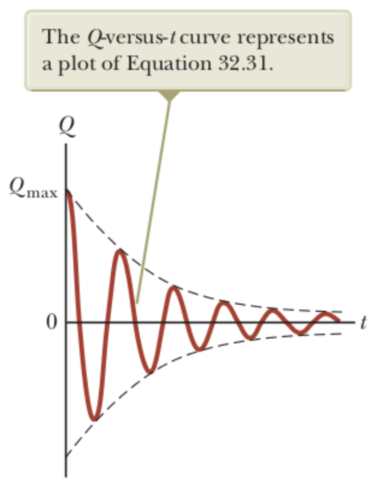
This equation is called the damped harmonic equation, whose solution is,

$$Q = Q_{max} e^{-Rt/2L} \cos(\omega_d t)$$

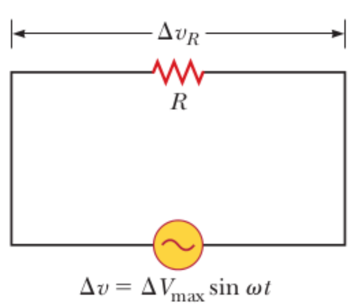
with

$$\omega_d = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

The circuit oscillates, but it also decays in time:



- As we saw previously, AC voltage supplies oscillate as: $\Delta v = \Delta V_{max} \sin \omega t$.
- A resistor supplied with AC voltage:



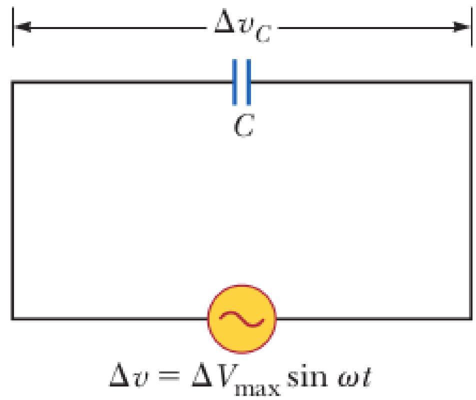
- Applying Kirchoff's loop rule: $\Delta v - i_R R = 0$

$$i_R = \frac{\Delta v}{R} = \frac{\Delta V_{max}}{R}\sin\omega t \equiv I_{max}\sin\omega t$$

Then the voltage drop across the resistor varies:

$$\Delta v_R = i_R R = I_{max} R \sin\omega t$$

- Inductors can be supplies with AC voltage:



with the loop rule:

$$\Delta v - L\frac{di_L}{dt} = 0$$

We rearrange to get,

$$di_L = \frac{\Delta V_{max}}{L}\sin\omega t dt \quad \rightarrow \quad i_L = -\frac{\Delta V_{max}}{\omega L}\cos\omega t$$

The inductor and applied AC current are out of phase by $\pi/2$ rad.

- Write

$$I_{max} = \frac{\Delta V_{max}}{\omega L}$$

- Call $X_L \equiv \omega L$ the **inductive reactance** then we have:

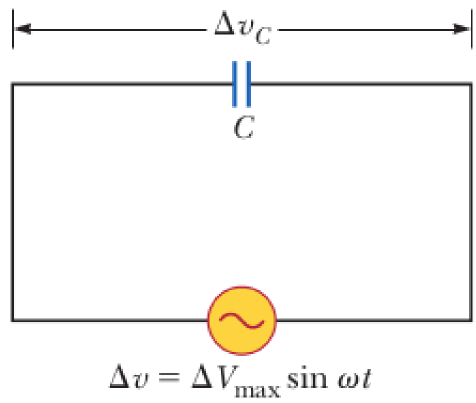
$$I_{max} = \frac{\Delta V_{max}}{X_L}$$

which puts it in a form comparable to $I = V/R$.

- Finally, we find that the voltage across the inductor is,

$$\Delta v_L = -L\frac{di_L}{dt} = -\Delta V_{max}\sin\omega t = -I_{max}X_L\sin\omega t$$

- Capacitors in an AC circuit:



- Using the same method of derivation as for the previous cases, we can find that the current in this circuit is,

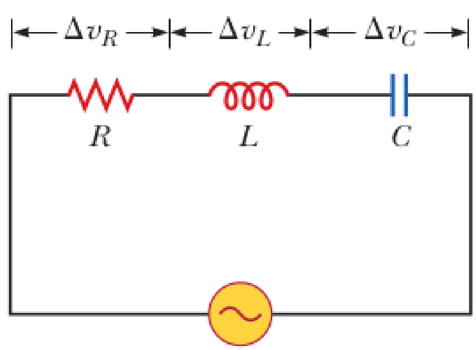
$$i_C = \omega C \Delta V_{max} \cos \omega t$$

we then have

$$I_{max} = \omega C \Delta V_{max} = \frac{\Delta V_{max}}{1/\omega C} \equiv \frac{\Delta V_{max}}{X_C}$$

where $X_C = \frac{1}{\omega C}$ is called the **capacitive reactance**.

- Finally, the RLC circuit:



- Refer to your book for the derivation, but for this circuit, we define the **impedence**

$$Z \equiv \sqrt{R^2 + (X_L - X_C)^2}$$

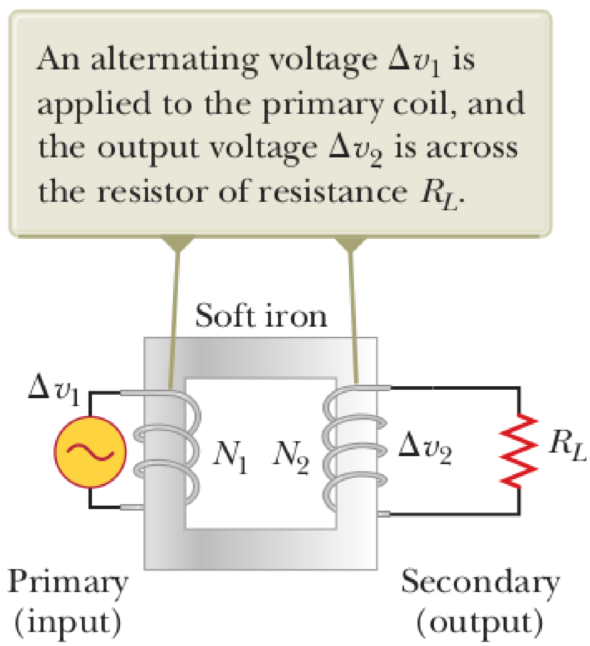
and

$$I_{max} = \frac{\Delta V_{max}}{Z}$$

with a phase angle between the current and voltage of:

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

- When $X_L > X_C$ (which occurs at high frequencies), the phase angle is positive, signifying that the current lags the applied voltage as in Active Figure 33.15b. We describe this situation by saying that the circuit is more inductive than capacitive. When $X_L < X_C$, the phase angle is negative, signifying that the current leads the applied voltage, and the circuit is more capacitive than inductive. When $X_L = X_C$, the phase angle is zero and the circuit is purely resistive.
- Because power is lost through resistance at a rate of I^2R, it is more efficient to transmit power at a very high voltage (thus lower current and less power loss).
- 350 kV lines are common, which are often brought down to 20,000 V at distributing stations, then 4,000 V before residential distribution, and then 120V and 240 V to customers. How is this accomplished?
- AC transformer:



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$$\Delta v_1 = -N_1 \frac{d\Phi_B}{dt}$$

$$\Delta v_2 = -N_2 \frac{d\Phi_B}{dt}$$

$$\Delta V_2 = \frac{N_2}{N_1} \Delta v_1$$