RC circuits (DC)

Nature is a curious force. We have already seen (in our homework) the most simplest case of combining resistors and capacitors. Now we are going to fully explore adding these together.

Lecture outline:

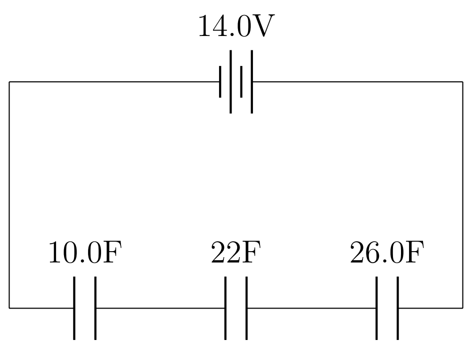
- Power in a circuit
- RC circuits

1 Guiding Question

How do we determine the power in a circuit? What happens when we combine a resistor and a capacitor in a circuit? What happens in various different configurations?

2 Review

Recall that the voltage drop across a capacitor is given by $Q/C = V$, and that in a circuit such as this:



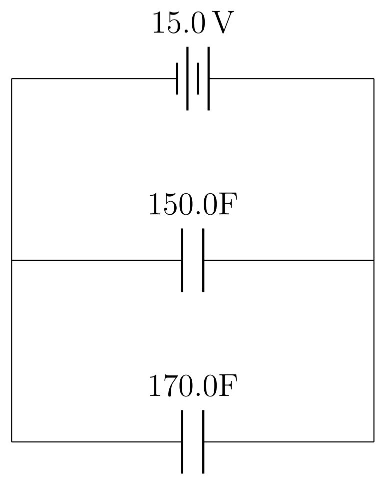
The loop rule gives us,

$$0 = 14.0\text{V} - \frac{Q}{22\text{F}} - \frac{Q}{10.0\text{F}} - \frac{Q}{26.0\text{F}} = 14.0\text{V} - Q\left(\frac{1}{22\text{F}} + \frac{1}{10.0\text{F}} + \frac{1}{26.0\text{F}}\right)$$

which tells us that for capacitors in series,

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots$$

On the other hand, in parallel:

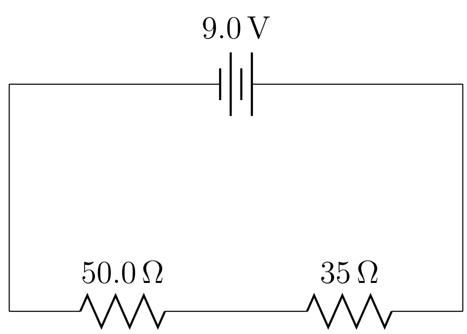


The voltage drop across each capacitor is the same, and so the total charge on both capacitors is

$$Q_{tot} = VC_1 + VC_2 = V(C_1 + C_2)$$

and so we see that in parallel, we simply add up the capacitors.

Resistors have the same pattern–though flipped. Consider the following circuit:



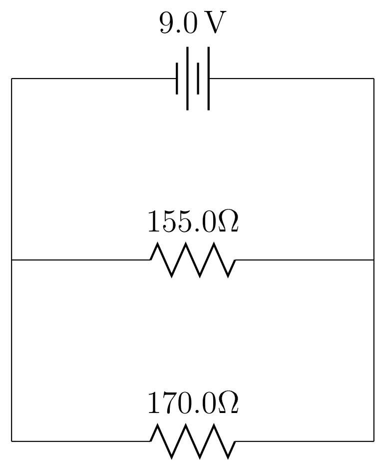
We learned that $V = IR$, and so the loop rule gives,

$$0 = 9.0\text{V} - I35\Omega - I50.0\Omega = 9.0\text{V} - I\,(35\Omega + 50.0\Omega)$$

and so we found that for series,

$$R_{eq} = R_1 + R_2 + \cdots$$

Considering a circuit such as:



We know that the current passing through the separate branches will not be the same unless the resistor is the same, so that we need to apply the junction rule:

$$I_{tot} = \frac{V}{R_1} + \frac{V}{R_2} = V\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{V}{R_{eq}}$$

and so we see that for resistors in parallel,

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots$$

3 Energy and Power in a circuit

3.1 Battery

The power supplied by a batter can be found by the following equation. Since the potential energy of a charge q across a voltage V is $U = qV$, then

$$P = \frac{dU}{dt} = \frac{dq}{dt}V = iV$$

For a battery, this is power pumped into the circuit.

3.2 Capacitors

For a capacitor, we can find the equation for stored (potential) energy from

$$dU = V\,dq = \frac{q}{C}dq \quad \rightarrow \quad \Delta U = \frac{1}{2}\frac{Q^2}{C}$$

Power is defined as the rate at which energy is consumed/transferred, has units $J/s = W$ Watts. For a normal capacitor, C doesn't change, and so,

$$P = \frac{d}{dt}\Delta U = \frac{q}{c}\frac{dq}{dt}$$

Since q isn't constant, we need to learn more about how the capacitor behaves in the circuit before getting more useful information out of this expression.

3.3 Resistors

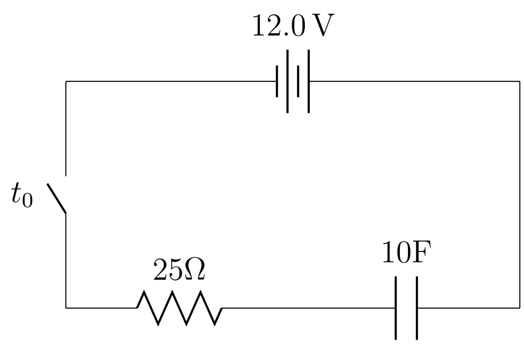
For resistors, $V = iR$, so the power dissipated by a resistor is:

$$P_r = iV_R = i^2 R$$

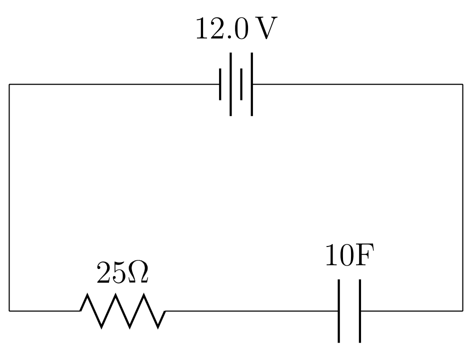
This energy is released to the universe as thermal energy (thus resistors feel warm to the touch and can burn or be burned).

4 The RC circuit

Consider the following circuit:



The switch is shown to be open, and this means the circuit is not activated–no current flows. Now the switch is activated:



What happens? Well, at first that question might be perplexing, but then we remember our physics training: let the concepts and laws of physics guide your solution. So let's apply the loop rule:

$$0 = \mathcal{E} - \frac{q}{C} - iR = \mathcal{E} - \frac{q}{C} - \frac{dq}{dt}R$$

where we have used the symbol $\mathcal{E}$ to represent the battery's voltage contribution, per convention. This equation is a differential equation, we simplify to:

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{\mathcal{E}}{R}$$

The RC denominator on the q suggest that the solution to q should take the form of an exponential such as $\pm e^{-t/RC}$ since,

$$\frac{d}{dt}e^{-t/RC} = \frac{e^{-t/RC}}{RC}$$

however, the fact that this must add up to a constant value, and the fact at time $t = 0$, $q = 0$ and at time $t \to \infty$ the capacitor should be $100\% = 1.0$ charged suggest instead a solution of the form $1 - e^{-t/RC}$ such that we still have,

$$\frac{d}{dt}(1 - e^{-t/RC}) = -\frac{e^{-t/RC}}{RC}$$

but the extra term will make our differential equation true. More generically, we choose the function:

$$q(t) = \zeta(1 - e^{-t/RC})$$

where ζ is some unknown constant. Plug this into the differential equation yields:

$$\begin{aligned} \frac{\zeta}{RC}e^{-t/RC} + \frac{\zeta}{RC}(1 - e^{-t/RC}) &= \frac{\mathcal{E}}{R} \\ \frac{\zeta}{RC} &= \frac{\mathcal{E}}{R} \\ \zeta &= \mathcal{E}C \equiv Q_0 \end{aligned}$$

and thus we have **the charge on a charging capacitor**:

$$q(t) = \mathcal{E}C(1 - e^{-t/RC}) \equiv Q_0\left(1 - e^{-t/RC}\right)$$

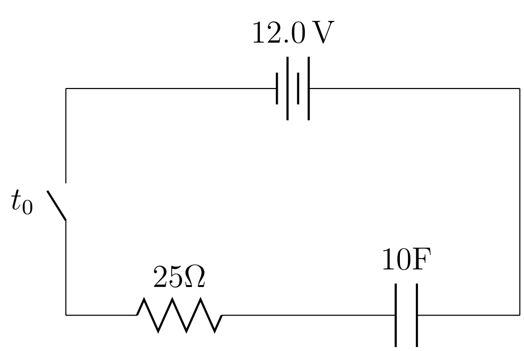
$$\boxed{q(t) = Q_0\left(1 - e^{-t/RC}\right)}$$

and the **current in an RC circuit with a charging capacitor** is:

$$i(t) = \frac{dq(t)}{dt} = \frac{\mathcal{E}C}{RC}e^{-t/RC} = \frac{\mathcal{E}}{R}e^{-t/RC} \equiv I_0 e^{-t/RC}$$

$$\boxed{i(t) = I_0 e^{-t/RC}}$$

Now suppose that a long, long time has passed (means the capacitor can be considered fully charged) and we open the switch again:



Now the differential equation takes the form:

$$\frac{dq}{dt} + \frac{q}{RC} = 0$$

This suggest that our solution might take the form:

$$q(t) = \gamma e^{-t/RC}$$

where γ is a constant we are to solve. We can confirm that this fits by taking the differential:

$$\frac{dq(t)}{dt} = -\frac{\gamma}{RC} e^{-t/RC} = -\frac{q(t)}{RC}$$

and knowing that at $t = 0$, the charge $q(t)$ is at its maximum and is the maximum charge the capacitor could hold at the voltage, i.e. $Q_0 = \mathcal{E}C$, **the charge on a discharging capacitor is:**

$$\boxed{q(t) = Q_0 e^{-t/RC}}$$

and **the current on a discharging RC circuit is:**

$$i(t) = -\frac{Q_0}{RC} e^{-t/RC} = -\frac{VC}{RC} e^{-t/RC} = -\frac{V}{R} e^{-t/RC} = -I_0 e^{-t/RC}$$

$$\boxed{i(t) = -I_0 e^{-t/RC}}$$

which is the same as for the case of charging, except in the opposite direction.

4.1 Energy conservation

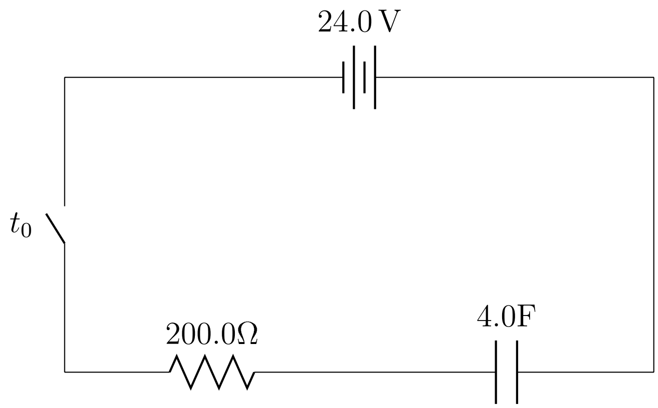
To charge the capacitor to a charge of Q_0, Q_0 charges must be pumped out of the batter. The battery does total work of $W = Q_0\mathcal{E} = C\mathcal{E}^2$. But we know that a fully charged capacitor stores $\frac{1}{2}Q^2/C = \frac{1}{2}C\mathcal{E}^2$. What happened to the other half of the batteries work? The power dissipated by the resistor is,

$$P_r = i^2(t)R$$

so the total energy "donated" to the universe by the resistor is,

$$\begin{aligned}
E_{th} &= \int P_r dt \\
&= \int i^2(t) R dt \\
&= \int_{t=0}^{\infty} \left(\frac{\mathcal{E}}{R} e^{-t/RC}\right)^2 R dt \\
&= \int_{t=0}^{\infty} \frac{\mathcal{E}^2}{R^2} e^{-2t/RC} R dt \\
&= \frac{\mathcal{E}^2}{R} \int_{t=0}^{\infty} e^{-2t/RC} dt \\
&= -\frac{\mathcal{E}^2}{R} \frac{RC}{2} e^{-2t/RC} \Big|_0^{\infty} \\
&= \frac{1}{2}\mathcal{E}^2 C
\end{aligned}$$

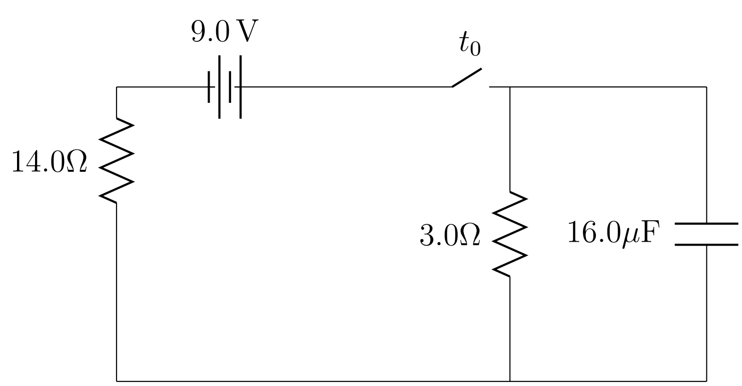
Half of the energy output produced by the battery is consumed by the resistor and converted into thermal energy, the other half is stored in the capacitor.



1. Find the charge, as a function of time, on the capacitor for the above circuit when the switch is closed. What is the value for each at 3 seconds in?

> **Solution:** The charge on the capacitor is given by,
>
> $$Q(t) = C\mathcal{E}\left(1 - e^{-t/RC}\right) = 9.6 \times 10^{1}\left(1 - e^{-t/\left(8.0\,\times\,10^{2}\right)}\right)$$
>
> at 3 seconds:
>
> $$Q(3) = C\mathcal{E}\left(1 - e^{-3/RC}\right) = 3.6 \times 10^{-1}\mathrm{C}$$

2. In the circuit below, find the current passing through each branch and the charge on the capacitor at the following times: once the switch is closed, and a long, long time later.



Solution: First start with the loop rule when the switch is first closed and there is zero charge on the capacitor:

$$9.0\text{V} - I14.0\Omega - \frac{0}{16.0\mu\text{F}} = 0$$

The trick here is knowing that the current will follow the path of least resistance, which in this case is the capacitor branch at $t = 0$. One could also apply the loop rule to the rightmost branch and find the current on the 3.0Ω branch is zero.

We solve for the largest-loop current to find:

$$I_0 = \frac{9.0\text{V}}{14.0\Omega} = 0.6\text{A}$$

After a very long time, the capacitor is fully charged and we simply have a resistor-in-series circuit with an equivalent resistance of $R_{eq} = 17.0\Omega$, and so the current passes only through the left-most loop and has the value:

$$I_\infty = \frac{V}{R_{eq}} = \frac{9.0\text{V}}{17.0\Omega} = 0.5\text{A}$$

At this point the capacitor has maximum charge, and we can apply the loop rule on the rightmost loop to find that

$$0 = I_\infty(3.0\Omega) - \frac{Q_\infty}{16.0\mu\text{F}}$$

$$Q_\infty = 2.5 \times 10^{-5}\text{F}$$