A COLLECTED DERIVATION OF STIRLING'S EQUATION

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ABSTRACT. Though this report presents an unoriginal derivation of Stirling's equation, it is useful to have it cogently presented.

1. Cowboy mathematics

As a physicist, one often disregards the formalities of mathematics which can hinder ones investigations just as often as they can help. In this section, we present an informal rough derivation of Stirling's formula. Consider that

$$\ln(n!) = \sum_{k=1}^{n} \ln(k). \tag{1.1}$$

Multiply both sides by Δk, so,

$$\lim_{\Delta k \to 0} \sum_{k=1}^{n} \ln(k) \Delta k = \int \ln(k) dk = k \ln(k) - k \tag{1.2}$$

Now of course, in our sum $\Delta k = 1$, but the relative size of Δk becomes smaller as $n \to \infty$, so if we write,and so we find loose enough reason to approximate, for $n \gg 1$,

$$\ln(n!) \approx n \ln(n) - n \tag{1.3}$$

2. The legal way I: buddy formulas and functions

Beyond undergraduate applications, the previous method does not provide a satisfactory understanding of Stirling's approximation. Following [1], we proceed to demonstrate a formal derivation.

2.1. Bernoulli Numbers.

We can define a set of numbers, the so called Bernoulli Numbers, via the equation,

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^2 \tag{2.1}$$

Complex analysis allows us to extract the $B_n \ni$

$$B_n = \frac{n!}{2\pi i} \oint_C \frac{1}{e^z - 1} \frac{dz}{z^n} \tag{2.2}$$

We can construct a contour which avoids the central pole (wrapping over the positive real axis, around a clockwise infinitesimal circle centered at the origin, and back along the positive real axis to rejoin the the main contour of clockwise orientation. We then have,

$$B_n = \frac{n!}{2\pi i} (-2\pi i) \sum_p Res[\pm p 2\pi i] \tag{2.3}$$

where

$$Res(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \tag{2.4}$$

With an application of l'Hopital's rule,

$$\lim_{z\to p2\pi i}\frac{z-p2\pi i}{e^z-1}=\lim_{z\to p2\pi i}\frac{1}{e^z}=1 \tag{2.5}$$

Thus we note that the odd residues beyond one will cancel each other out

$$B_n=\frac{n!}{2\pi i}(-2\pi i)2\sum_{p=1}^{\infty}\frac{1}{p^n(2\pi i)^n}=-\frac{(-1)^{n/2}2n!}{4\pi^2}\left(\sum_{p=1}^{\infty}p^{-n}=\zeta(n)\right) \tag{2.6}$$

For example, $B_0=1, B_1=-1/2, B_2=1/6, B_3=0, B_4=-1/30$, and so on. Bernoulli functions are derived in the same way, defined by

$$\frac{xe^{xs}}{e^x-1}=\sum_{n=0}^{\infty}B_n(s)\frac{x^n}{n!} \tag{2.7}$$

This is easily solved by expanding the extra e^{xs} factor and relating the results to the previous results for the Bernoulli numbers, i.e.,

$$B_n(s)=\frac{n!}{2\pi i}\oint\frac{e^z}{e^z-1}\frac{dz}{z^n}=\frac{n!}{2\pi i}\oint\frac{1}{e^z-1}\left(\frac{1}{z^n}+\frac{s}{z^{n-1}}+\frac{s^2}{2!z^{n-2}}+\cdots\right)dz \tag{2.8}$$

For example,

$$B_3(s)=B_3+B_2\frac{3!}{2!}s+B_1\frac{3!}{1!2!}s^2+s^3\frac{3!}{0!3!}B_0=s^3-\frac{3}{2}s^2+\frac{1}{2}s \tag{2.9}$$

2.2. **Euler-Maclaurin Integration Formula.** To start,

$$\int_0^1 f(x)dx=\int_0^1 f(x)B_0(x)dx \tag{2.10}$$

It is also obvious that $B_1'(x)=B_0(x)=1$ so that,

$$\int_0^1 f(x)dx=f(1)B_1(1)-f(0)B_1(0)-\int_0^1 f'(x)B_1(x)dx \tag{2.11}$$

Carrying on and seeing that $B_{2n}(1)=B_{2n}(0)=B_2n$ and $B_{2n+1}(1,0)=0$, we would find,

$$\int_0^1 f(x)dx=\frac{1}{2}[f(1)+f(0)]-\sum_{p=1}^{q}\frac{1}{(2p)!}B_{2p}[f^{(2p-1)}(1)-f^{(2p-1)}(0)]+\frac{1}{(2q)!}\int_0^1 f^{(2q)}(x)B_{2q}(x)dx \tag{2.12}$$

The transform $x\to x+1$, $x+1\to x+2$,etcetera, yields,

$$\int_0^n f(x)dx = \frac{1}{2}\left(f(0)+f(n)\right)+\sum_{k=1}^{n-1}f(k) \tag{2.13}$$

$$-\sum_{p=1}^{q}\frac{1}{(2p)!}B_{2p}[f^{(2p-1)}(1)-f^{(2p-1)}(0)]+\frac{1}{(2q)!}\int_0^1 B_{2q}(x)\sum_{\chi=0}^{n-1}f^{(2q)}(x+\chi)dx$$

2.3. **Gamma, Digamma, Polygamma Functions.**

$$z!=z\Gamma(z)=\lim_{n\to\infty}\frac{n!}{(z+1)(z+2)\cdots(z+n)}n^2 \tag{2.14}$$

$$\ln(z!)=\lim_{n\to\infty}\left(ln(n!)+z\ln(n)-\sum_{k=1}^{n}ln(z+k)\right)\ni \tag{2.15}$$

$$\frac{d}{dz}\ln(z!)\equiv F(z)=\lim_{n\to\infty}\left(ln(n)-\sum_{k=1}^{n}\frac{1}{z+k}\right) \tag{2.16}$$

2.4. **Beta Function and the Legendre Duplication Formula.** Consider the fact that,

$$m!n! = \lim_{a^2\to\infty} \int_0^{a^2} e^{-u}u^m du \int_0^{a^2} e^{-v}v^n dv \tag{2.17}$$

When we transform to polar coordinates, we have,

$$m!n! = (m+n+1)!2\int_0^{\pi/2} \cos^{2m+1}\theta \sin^{2n+1}\theta d\theta \tag{2.18}$$

The beta function is so defined as,

$$B(m+1,n+1) \equiv 2\int_0^{\pi/2} \cos^{2m+1}\theta \sin^{2n+1}\theta d\theta = \frac{m!n!}{(m+n+1)!} \tag{2.19}$$

With $t = \cos^2\theta$, we have

$$\frac{m!n!}{(m+n+1)!} = \int_0^1 t^m(1-t)^n dt \underbrace{=}_{t=x^2} 2\int_0^1 x^{2m+1}(1-x^2)^n dx \tag{2.20}$$

From these we can obtain the Legendre duplication formula, where we start with,

$$\frac{z!z!}{(2z+1)!} = \int_0^1 t^z(1-t)^z dt \underbrace{=}_{t=(1+s)/2} 2^{-2z}\int_0^1 (1-s^2)^z ds \tag{2.21}$$

From the definition of the Beta function, we have,

$$\frac{z!z!}{(2z+1)!} = 2^{-2z-1}\frac{z!(-\frac{1}{2})!}{(z+\frac{1}{2})!} \tag{2.22}$$

It is easy to show that $(-\frac{1}{2})! = \sqrt{\pi}$. Multiplying through equation 2.22 by $z+\frac{1}{2}$, we have,

$$z!(z-\frac{1}{2})! = 2^{-2z}\sqrt{\pi}(2z)! \tag{2.23}$$

3. The legal way II: taking it home

The application of the Euler-Maclaurin Integration Formula to the function

$$\int_0^\infty \frac{dx}{(z+x)^2} = \frac{1}{z}$$

and recalling the Polygamma function, we know that,

$$\frac{1}{z} = \frac{1}{2z^2} + F^1(z) - \sum_{n=1}^\infty \frac{B_2n}{z^{2n+1}} \tag{3.1}$$

Which is to say,

$$\frac{d}{dz}F(z) = \frac{1}{z} - \frac{1}{2z^2} + \sum_{n=1}^\infty \frac{B_2n}{z^{2n+1}} \tag{3.2}$$

Which is also to say, after some integration,

$$ln(z!) = \aleph + \left(z+\frac{1}{2}\right)\ln z - z - \int\sum_{n=1}^\infty \frac{B_{2n}}{2nz^{2n}}dz \tag{3.3}$$

We can solve for $\aleph$ by using the Legendre duplication formula in 3.3 and taking $z \to \infty$. In this sense, we will proceed but ignore all pieces on the order of z^{-1} or less, so that,

$$
\begin{aligned}
ln(z!) + ln((z - \frac{1}{2})!) &= -2z\ln(2) + \frac{1}{2}ln(\pi) + ln((2z)!) \\
&= 2\aleph + \left(z + \frac{1}{2}\right)\ln(z) - z + \cdots + z\ln(z(1 - \frac{1}{2z})) - z + \frac{1}{2} + \cdots \\
&= 2\aleph + \left(2z + \frac{1}{2}\right)\ln(z) - 2z + \cdots \\
&= -2z\ln(2) + \frac{1}{2}\ln(\pi) + \aleph + \left(2z + \frac{1}{2}\right)\ln(2z) - 2z + \cdots
\end{aligned}
$$

Here we must detail the disappearance of the 1/2:

$$
z\ln(1 - \frac{1}{2z}) = z\left(\frac{-1}{2z} - \frac{1}{2!}\left(\frac{-1}{2z}\right)^2 + \cdots\right) \underbrace{\to}_{z\to\infty} -\frac{1}{2} \tag{3.4}
$$

Now we can cancel out what we may cancel out and separate $\ln(2z) = \ln(2) + \ln(z)$, finally taking the limit of z,

$$
\aleph = \frac{1}{2}\ln(2) + \frac{1}{2}\ln(\pi) = \frac{1}{2}\ln(2\pi) \tag{3.5}
$$

At last, we have

$$
\ln(z!) = \frac{1}{2}\ln(2\pi) + \left(z + \frac{1}{2}\right)\ln(z) - z + \frac{1}{12z} - \frac{1}{360z^3} + \cdots \tag{3.6}
$$

Finally, by noting for example that,

$$
e^{1/12z} = 1 + \frac{1}{12z} + \frac{1}{2!}\left(\frac{1}{12z}\right)^2 + \cdots \tag{3.7}
$$

we also have,

$$
z! = \sqrt{2\pi}z^{z+1/2}e^{-z}\left(1 + \frac{1}{12z} + \frac{1}{288z^2} + \cdots\right) \tag{3.8}
$$

References

[1] G. Arfken, H. Weber, *Mathematical Methods for Physicists*, Harcourt Academic Press, 2001

[2] E. Saff, A. Snider, *Fundamentals of Complex Analysis for Mathematics, Science, and Engineering, 2nd ed.*, Prentice Hall, 1993