THE PFAFFIAN AND THE WEDGE PRODUCT

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The following problem demonstrates the relation of the Pfaffian with the wedge product: Find an $\omega \in Alt^2\mathbb{R}^4 \ni \omega \wedge \omega \neq 0$

Based on definitions of the wedge product,

$$\omega = \zeta_{ij} e^i \wedge e^j \quad i, j = 0 \cdots 3$$

we have

$$\omega \wedge \omega = \left(\zeta_{ij} e^i \wedge e^j\right) \wedge \left(\zeta_{ab} e^a \wedge e^b\right) = \left(\sum_{\sigma \in S(p,q)} \operatorname{sign}(\sigma) \zeta_{\sigma(i)\sigma(j)} \zeta_{\sigma(a)\sigma(b)} \right) e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

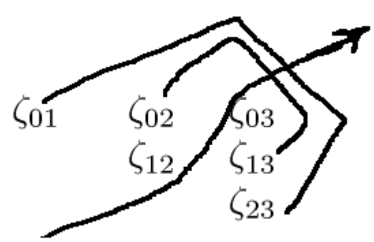
Combination	Sign
(0,1)(2,3)	+ 1
(0,2)(1,3)	-1
(0,3)(1,2)	+1
(1,2)(0,3)	+1
(1,3)(0,2)	-1
(2,3)(0,1)	+1

One explicates this and finds,

$$\omega \wedge \omega = \left(2\zeta_{01}\zeta_{23} - 2\zeta_{02}\zeta_{13} + 2\zeta_{03}\zeta_{12}\right) e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

We might stop here, and simply search for a combination of coefficients so that the above product is not zero, but we must note that the combination is suggestive of Sarrus's scheme in that the ζ are multiplied with members of their same diagonal, via Sarrus's scheme, but with the negative counterparts not included in the multiplication. Consider:

$$\hat{\zeta} = \begin{pmatrix} 0 & \zeta_{01} & \zeta_{02} & \zeta_{03} \\ -\zeta_{01} & 0 & \zeta_{12} & \zeta_{13} \\ -\zeta_{02} & -\zeta_{12} & 0 & \zeta_{23} \\ -\zeta_{03} & -\zeta_{13} & -\zeta_{23} & 0 \end{pmatrix}$$



The mechanisms of the wedge product prefactor coupling and the determinant formation are suggestively similar. It is one of the greatest sadnesses in mathematics that Sarrus's scheme does not extend beyond three dimensional arrays. The determinant is instead found by Laplace's theorem, as

$$\begin{vmatrix} 0 & \zeta_{01} & \zeta_{02} & \zeta_{03} \\ -\zeta_{01} & 0 & \zeta_{12} & \zeta_{13} \\ -\zeta_{02} & -\zeta_{12} & 0 & \zeta_{23} \\ -\zeta_{03} & -\zeta_{13} & -\zeta_{23} & 0 \end{vmatrix} = (-1)^{(1+3)}\zeta_{02}\begin{vmatrix} -\zeta_{01} & 0 & \zeta_{13} \\ -\zeta_{02} & -\zeta_{12} & \zeta_{23} \\ -\zeta_{03} & -\zeta_{13} & 0 \end{vmatrix} + \cdots =$$

$$(\zeta_{02}^2\zeta_{13}^2 - \zeta_{12}^2\zeta_{03}^2 + \zeta_{23}^2\zeta_{01}^2) - 2\zeta_{03}\zeta_{12}\zeta_{13}\zeta_{02} - 2\zeta_{23}\zeta_{01}\zeta_{13}\zeta_{02} + 2\zeta_{23}\zeta_{03}\zeta_{01}\zeta_{12} =$$
$$= (\zeta_{01}\zeta_{23} - \zeta_{02}\zeta_{13} + \zeta_{03}\zeta_{12})^2$$

Does this then suggest that

$$det(\hat{\zeta}) = \sum_{\sigma\in S(p+q)} \operatorname{sign}(\sigma)\prod_{i=0}^{p+q}\hat{\zeta}_{i\sigma(j)} = \left(\frac{1}{p!q!}\sum_{\sigma\in S(p+q)}\operatorname{sign}(\sigma)\zeta_{\sigma(i)\sigma(j)}\zeta_{\sigma(a)\sigma(b)}\right)^2 \; ?$$

At this point, we investigated further and found a reference on Wikipedia (of course) regarding the Pfaffian in relation to the wedge product. We also consulted with our colleague Daniel Cross who confirmed that this was indeed the right path. Further investigation brought us into consideration of the extensive work on determinants by Thomas Muir[1], and we wish to take the reader along this path to show how indeed the above equation is true via Muir's work.

A determinant can be defined by use of minors, e.g.

$$\Delta = (-1)^{p+1}a_{p1}A_{p1} + (-1)^{p+2}a_{p2}A_{p2} + \cdots + (-1)^{n+p}a_{pn}A_{pn}$$

If we switch $a_{pr} \to a_{qr}$, then the determinant of such will have two identical rows and thus be zero. This allows us to find that when we multiply a matrix by its adjugate,

$$\det(A|A_{ij}|) = \begin{vmatrix} A & 0 & 0 & \cdots & 0 \\ 0 & A & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & A \end{vmatrix} = |A|^n$$

Thus the determinant of the adjucate is the determinant of A^{n-1}. Now to get to our next point, we have found the fastest path is a short proof by S. Parameswaran published in *The American Mathematical Monthly* (Vol. 61, No. 2. (Feb., 1954), p. 116). We recreate the proof in whole here. For n even, we define the determinant of our matrix to be,

$$\Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Then Δ_{n-2} can be defined as the minor of $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$. Parameswaran next notes that,

$$\begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \times \begin{vmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ A_{13} & A_{23} & \cdots & A_{n3} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{vmatrix} = \begin{vmatrix} A_{11} & A_{21} \\ A_{12} & a_{22} \end{vmatrix}\Delta_n^{n-2}$$

$$\Delta_{n-2}\Delta_n^{n-1} = \begin{vmatrix} A_{11} & A_{21} \\ A_{12} & a_{22} \end{vmatrix}\Delta_n^{n-2}$$

[1] *A Treatise on the Theory of Determinants*

We now rejoin Muir where we write this generically as,

$$A_{rr}A_{ss} - A_{rs}A_{sr} = \Delta_{rs,rs}\Delta_n$$

For zero-axial skew-symmetric determinants, the left side simplifies to A_{rs}^2. With use of this identity, we can apply Laplace's theorem again and write that

$$\begin{aligned}\Delta^{1/2} &= (-1)^r \left\{ a_{r1}A_{1r,1r}^{1/2} - a_{r2}A_{2r,2r}^{1/2} + \cdots \right\} \\ &\equiv ff(a_1 \quad 2n) =^{|} |a_1 \quad 2n| = \left| \begin{array}{cccc} a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{23} & \cdots & a_{2n} \\ & & a_{3n} & \cdots \\ & & & \cdots \end{array} \right|\end{aligned}$$

The latter three being various notations, according to Muir, for the Pfaffian (though Wikipedia suggest the modern notation is simply Pf()). To bring this all home, we finally note that for our given four dimensional array,

$$(-1)^r \left\{ a_{r1}A_{1r,1r}^{1/2} - a_{r2}A_{2r,2r}^{1/2} + \cdots \right\} =$$

$$\zeta_{01}\sqrt{\begin{vmatrix} 0 & \zeta_{23} \\ -\zeta_{23} & 0 \end{vmatrix}} - \zeta_{02}\sqrt{\begin{vmatrix} 0 & \zeta_{13} \\ -\zeta_{13} & 0 \end{vmatrix}} + \zeta_{03}\sqrt{\begin{vmatrix} 0 & \zeta_{12} \\ -\zeta_{12} & 0 \end{vmatrix}} =$$

$$\zeta_{01}\zeta_{23} - \zeta_{02}\zeta_{13} + \zeta_{03}\zeta_{12}$$

This confirms our suspicion, and generically we write[2]:

$$\frac{\omega^n}{n!} = \mathrm{ff}(\hat{\zeta})e^1 \wedge e^2 \wedge \cdots \wedge e^{2n}$$

An example of such a matrix would be the electromagnetic variable matrix seen in electrodynamics[3].

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[2]See for example http://en.wikipedia.org/wiki/Pfaffian

[3]See for example page 60 of Greg Naber's *Topology, Geometry, and Gauge Fields: Interactions*