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IS QUANTUM DECOHERENCE THE VON NEUMANN WAVE COLLAPSE?

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ABSTRACT. How long does it take for a wave function to collapse? This question, asked by a skeptical undergraduate many years ago, was never properly answered by his unfortunate professor. And so, many years later, the question stands.

The goal of this report is to explore this question as modern theory might approach it. We will find that the subtext of the question, a profound skepticism about the postulate of wave-collapse, remains unanswered.

The theoretical program of decoherence can "effectively" replace the wave-collapse and even tell us how long the transition from the quantum realm to the classical realm will take. Though we are still left with a mystery, it is a mystery which has been pushed off the table in regards to experimental physics.

Contents

1. What is Decoherence?

It is best to introduce decoherence with a simple example. Before doing so, we need to have something to decohere. The job of decoherence is to bring a quantum system into an apparently classical state. What especially differentiates a quantum system from a classical system is the concept of a superposition of states. In the classical realm of physics, we would say that a particle is at a position (x,y,z). In the quantum realm, the formalism allows us to state that a particle is in a superposition of positions (say, (x_1, y_1, z_1) and (x_2, y_2, z_2)). However, as the canon of quantum mechanics postulates, when we actually measure the position of this particle, we will find it at either of the two positions, that is, we will have *"collapsed the wave function"* into one or the other state. A classical mind (which we all have) would ask, what does this mean? This is the so called "measurement problem". David Albert puts it better when he writes,

> The dynamics and the postulate of collapse are flatly in contradiction with one another ... the postulate of collapse seems to be right about what happens when we make measurements, and the dynamics seems to be bizarrely wrong about what happens when we make measurements, and yet the dynamics seems to be right about what happens whenever we aren't making measurements. [2]

Let us be more explicit about what he means here. It was von Neumann [12] who first articulated the so called "dynamical dualism" that haunted the original formulations of quantum mechanics, though Bohr touched on the issue earlier in proposing the "quantum leap" into states which, like the wave collapse, is proposed as a dynamically discontinuous process. Primarily, the evolution of a quantum system is described by the Shrödinger equation,

$$(1.1) \qquad i\hbar \frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle,$$

applying to a system under isolation. There is also the evolution that occurs due to measurement, and this is the infamous collapse:

$$(1.2) \qquad |\psi\rangle = \sum_n c_n |n\rangle \longrightarrow |n_i\rangle$$

Technically, Equation 1.2 is called by von Neumann the "first intervention" and Equation 1.1 is the "second intervention." The first intervention is what we will identify as the wave collapse. It describes a superposition of states suddenly 'collapsing' into one eigenstate, the measurement.

If theoreticians could provide a mechanism within the context of quantum mechanics in which the evolution superficially described by von Neumann's first intervention is achieved, without the need for an ad hoc mechanism or the insistence that we live in a universe in which "consciousness or spirit...play an important and fundamental role" in physical phenomenon [16], they will have made great progress in solidifying quantum mechanics (even further) as a sufficient theory. By this we mean that the protests that quantum mechanics is somehow incomplete (e.g. the EPR paradox [6]) can be better addressed (beyond Bell's theorem).

Given the substantial success of quantum mechanics in correctly predicting the outcome of every experiment thus far conceived and executed to test it, we would prefer that any mechanism designed to describe what occurs when a measurement is taken does so in the context of standard quantum mechanics, which is to say, does not require a modification of the Shrödinger equation.

The alternative is to accept a modification of the standard formulation of quantum mechanics. There are many various "collapse models" one could consider ([10], see also [11] as a short, reasonable example). And though it serves a physicist well to

consider fresh, new ideas, it also serves us well to realize that such ideas are almost always wrong. In any case, these alternatives are not the focus of this report.

Decoherence offers a theoretical framework in which the measurement problem can be swept under the carpet (pushed into a system larger than that which we can observe). The effect is that quantum mechanics can be studied and presented to a student without the need for the ad hoc "wave collapse" being presented as a primary tool of the theory. One can achieve, in many cases, the same apparent effect of a wave collapse without recourse to von Neumann's mysterious first intervention.

Thus we clarify that decoherence is not a new theory unto itself, but is instead an efficient and fruitful repackaging of theory. It does not solve the measurement problem, and most certainly wouldn't have satisfied the reservations of Einstein in his later years. Nevertheless, given its elegance in providing an apparent transition from the quantum realm to the classical realm, and its experimental success, we believe the time has come that decoherence be incorporated into graduate level quantum mechanics courses. This report is designed to be a self-contained introduction to the topic appropriate for a graduate student.

Our presentation will be fairly more detailed than that typically found in the literature [8, 9] and by necessity will serve as minor introduction to quantum optics (the reader is also referred to the appendix as needed).

2. Theoretical Example of Superposition

Unique to wave theories is the concept of superposition of states. Indeed, the interference of states is often used as a defining characteristic of quantum systems such that the destruction of a pre-existing interference pattern is identified with a transition into the classical realm ("the appearance of the classical world" [15]).

In demonstrating the mathematics of decoherence, the simplest system we could possibly start with must entail at least two states capable of superposition. We consider a simple harmonic oscillator in the state,

$$(2.1) \qquad |\psi\rangle = |\alpha_1\rangle + |\alpha_2\rangle$$

Generically, we define α as an eigenvalue of the annhilation operator state of the oscillator,

$$(2.2) \qquad |\alpha\rangle = \exp(-\frac{|\alpha|^2}{2}) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

These so called coherent states represent a quantum system which is very close to being in a classical state (see appendix). Let us ignore, for now, the need for normalization (or if preferred, assume the system is normalized as is). It is well known that the Hamilton for such a system is simply given by

$$H = \hbar\omega a^\dagger a$$

The unitary evolution of this wave function is,

$$(2.3) \qquad |\psi(t)\rangle = \exp(-i\omega a^\dagger a t)(|\alpha_1\rangle + |\alpha_2\rangle)$$

Note that

$$\exp(-i\omega a^\dagger a t)\exp(-\frac{|\alpha|^2}{2}) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \exp(-\frac{|\alpha|^2}{2}) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \exp(-i\omega t n)|n\rangle$$

4

It can be written,

$$(2.4) \qquad |\psi(t)\rangle = (|\alpha_1 \exp(-i\omega t)\rangle + |\alpha_2 \exp(-i\omega t)\rangle)$$

The density matrix is

$$(2.5) \qquad \rho(t) = \sum_{i,j=1}^{2} |\alpha_i \exp(-i\omega t)\rangle\langle\alpha_j \exp(-i\omega t)\rangle|$$

To get a clear picture of the interference this superposition entails, we can take a look at this system in the coordinate representation. We have that,

$$a|\alpha\rangle = \alpha|\alpha\rangle = \frac{1}{\sqrt{2\hbar\omega}}(\omega q + ip)|\alpha\rangle$$

We can thus obtain,

$$(2.6) \qquad \langle q'|\alpha\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{\omega}{2\hbar}q'^2 + \sqrt{\frac{2\omega}{\hbar}}\alpha q' - \frac{|\alpha|^2 + \alpha^2}{2}\right)$$

Let $|\alpha_i \exp(-i\omega t)\rangle \equiv |\alpha_i(t)\rangle$. Then we can write,

$$\langle q'|\rho(t)|q'\rangle = \langle q'|\left(\sum_{i,j}|\alpha_i\rangle\langle\alpha_j|\right)|q'\rangle$$

$$(2.7) \qquad \langle q'|\rho(t)|q'\rangle = |\langle q'|\alpha_1(t)\rangle|^2 + |\langle q'|\alpha_2(t)\rangle|^2 + 2Re\langle q'|\alpha_1(t)\rangle\langle\alpha_2(t)|q'\rangle$$

The latter follows from the fact that the two cross terms will be conjugates, and for any complex number z, $z + z^* = 2Re(z)$. We can partition this function into real and imaginary parts as follows. We write

$$\alpha(t) = \alpha \exp(-i\omega t) = \alpha(\cos\omega t - i\sin\omega t)$$

We note that

$$
\begin{aligned}
\frac{\alpha^2 + |\alpha|^2}{2} &= \frac{\alpha(\cos^2\omega t - \sin^2\omega t - 2i\cos\omega t\sin\omega t + \cos^2\omega t + \sin^2\omega t)}{2} \\
(2.8) \qquad &= \alpha^2\left(\cos^2\omega t - \frac{i}{2}\sin 2\omega t\right)
\end{aligned}
$$

Thus,

$$\langle q'|\alpha\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\left(\sqrt{\frac{\omega}{2\hbar}}q' - \alpha\cos\omega t\right)^2\right) \exp i\left(-\sqrt{\frac{2\omega}{\hbar}}\alpha q'\sin\omega t + \frac{\alpha^2}{2}\sin 2\omega t\right)$$

It is now conventional to chose a simple case in which $\alpha_1 \equiv \alpha$ and $\alpha_2 \equiv -\alpha$,

$$(2.9) \qquad |\langle q'|\alpha_{1,2}\rangle|^2 = \sqrt{\frac{\omega}{\pi\hbar}} \exp\left(-\left(\sqrt{\frac{\omega}{\hbar}}q' \pm \sqrt{2}\alpha\cos\omega t\right)^2\right) \equiv I_{1,2}^2$$

The convenient $\pm\alpha$ form makes calculation of the cross term easy, and it is,

$$(2.10) \qquad 2Re\langle q'|\alpha_1(t)\rangle\langle\alpha_2(t)|q'\rangle = 2I_1 I_2 \cos\left(2\sqrt{\frac{2\omega}{\hbar}}\alpha q'\sin\omega t\right) \equiv 2I_1 I_2 \cos\theta(t)$$

$$(2.11) \qquad \langle q'|\rho(t)|q'\rangle = \overbrace{I_1^2 + I_2^2}^{\aleph\ \text{classical}} + \overbrace{2I_1 I_2 \cos\theta(t)}^{\chi\ \text{quantum interference}}$$

If Decoherence is to bring us into the classical realm, χ must be destroyed. Let us be more explicit here. Equation 2.11 gives us only partial information, i.e. the

probabilities for any given position q'. To represent the density matrix in proper form, we need integrate over all positions, i.e. the density matrix is

$$(2.12) \qquad \int_{-\infty}^{\infty} \begin{pmatrix} I_1^2 & I_1 I_2 \cos\theta(t) \\ I_2 I_1 \cos\theta(t) & I_2^2 \end{pmatrix} dq'$$

A classical density matrix would not have off diagonal terms. The total probability of finding the particle in one state or the other is given by

$$(2.13) \quad Tr\left(\int_{-\infty}^{\infty} \begin{pmatrix} I_1^2 & I_1 I_2 \cos\theta(t) \\ I_2 I_1 \cos\theta(t) & I_2^2 \end{pmatrix} dq' \right) = \int_{-\infty}^{\infty} \left(I_1^2 + I_2^2 \right) dq' = 1$$

Experimental demonstration of the effect of superposition has made much progress in the last decade, and we discuss one such example in the next section.

3. Experimental Example of Superposition

A review of an experiment in which superposition is demonstrated follows. We focus specifically on work done by the Laboratoire Brossel in Paris [3, 4, 1]. In their experiments using Ramsey interferometry, a microwave cavity, and an ensemble of Rydberg atoms, the Paris team not only demonstrate quantum superposition, but they were also able to demonstrate decoherence. The reader curious for more detail than presented here is especially encouraged to read the theoretical proposal of Davidovich et al. [3].

The experimental set up is shown in Figure 3.1. A resonant cavity (C) is prepared to contain a field off resonance with the $|g\rangle \rightarrow |e\rangle$ resonance , though for the following two examples this cavity is inactive (but plays a crucial part in the decoherence experiment). The Paris group uses a high-Q cavity, where $Q = 2\pi\nu\varepsilon/P$, is a measure of the efficiency of the cavity in storing a field (ε is the energy stored, ν is the resonant frequency, and $P = -dE/dt$ is the energy loss) such that the average lifetime of a resonant photon in the cavity is proportional to Q [17] and thus has a relatively long relaxation time. A coherent state is injected from (S) into the cavity (C) in a later experiment discussed regarding decoherence.

Two low-Q cavities are used (R1, R2) to apply microwave fields produced by S'. B represents a "black-box" which prepares Rydberg atoms (atoms with one valence electron in an extremely high n state), and (De) and (Dg) are detectors which provide an electric field that is sufficient to ionize the Rydberg atoms in their excited state (De) or ground state (Dg). Laser beams L1, L1' are used to select for atoms of the proper velocity, and the L2, B setup excites Rydberg atoms into their circular states (high quantum number n). The entire setup is enclosed and cooled to 0.6K to make thermal radiation negligible.
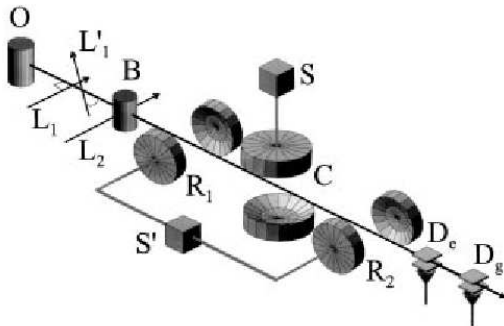

Figure 3.1. Experimental setup from [1] Paris group (1996)

Henceforth we shall assume ideality and ignore experimental details which detract from the flow of our report. The actual experiment followed closely. The Rydberg atoms are excited into the states $|e\rangle$ or $|g\rangle$ corresponding to quantum number $n = 51$ and $n = 50$ respectively. The $|e\rangle \rightarrow |g\rangle$ transition frequency is 51.099 GHz [3], and cavities R1 and R2 contain fields resonant with this transition.

To describe what happens to the atom as it passes through R1 toward C and R2, we need to introduce (briefly) the semi-classical Rabi model [7] appropriate for such resonance. The derivation is simpler in the semi-classical form, but similar results can be obtained (see appendix) in the full quantum (i.e. Jaynes-Cummings model) derivation. Furthermore, it can be demonstrated that the fields in R1 and R2 behave classically and so do not produce entanglement, thus justifying a semi-classical approach [18]. We assume an atom capable of being in two states (a good approximation for a Rydberg atom under controlled circumstances), $|e\rangle$ and $|g\rangle$ where $\omega_0 = (E_e - E_g)/\hbar$ and the laser field produces a frequency $\omega \approx \omega_0$. We assume an interaction Hamiltonian of simply $H_i(t) = V_0 \cos \omega t$. The state vector can be written,

(3.1) $\qquad |\psi(t)\rangle = C_g(t) \exp(-iE_g t/\hbar)|g\rangle + C_e(t) \exp(-iE_e t/\hbar)|e\rangle$

Using the time-dependent Schrödinger equation gives

$$
\begin{aligned}
\zeta &= \langle e|V|g\rangle \\
\dot{C}_g &= -\frac{i}{\hbar}\zeta \cos \omega t \exp(-i\omega_0 t)C_e \\
\dot{C}_e &= -\frac{i}{\hbar}\zeta \cos \omega t \exp(-i\omega_0 t)C_g
\end{aligned}
$$

A valid assumption has us suppose that $C_g(0) = 1$ and $C_e(0) = 0$, and using the identity $\cos \omega t = (e^{i\omega t} + e^{-i\omega t})/2$, we make application of the "Rotating wave approximation" where terms $\omega + \omega_0$ are disregarded and terms $\omega - \omega_0$ are kept (since ω is in near resonance, this latter term dominates the behavior),

$$
\begin{aligned}
\dot{C}_g &= -\frac{i}{2\hbar}\zeta \exp(-i\Delta t)C_e \\
\dot{C}_e &= -\frac{i}{2\hbar}\zeta \exp(+i\Delta t)C_g \\
0 &= \ddot{C}_e - i\Delta \dot{C}_e + \frac{\zeta^2 C_e}{4\hbar^2}
\end{aligned}
$$

Here $\Delta = \omega_0 - \omega$ is called the detuning of the atomic transition frequency and laser field. After assuming a characteristic solution of $\exp(i\lambda t)$ we arrive at a characteristic equation of $\lambda_\pm = (\Delta \pm \sqrt{\Delta^2 + \zeta^2/\hbar^2})/2$, and fitting the given initial conditions, one finds,

$$
\begin{aligned}
C_e(t) &= A_+ \exp(i\lambda_+ t) + A_- \exp(i\lambda_- t) \\
A_\pm &= \mp \frac{\zeta}{2\hbar}\frac{1}{\sqrt{\Delta^2 + \eta^2/\hbar^2}} \\
\Omega_R &= \sqrt{\Delta^2 + \zeta^2/\hbar^2} \\
C_e(t) &= i\frac{\zeta}{\Omega_R \hbar}\exp(i\Delta t/2) \\
C_g(t) &= \exp(i\Delta t/2)\left(\cos(\Omega_R t/2) - i\frac{\Delta}{\Omega_R}\sin(\Omega_R t/2)\right)
\end{aligned}
$$

To introduce the concept of $\pi/2$ and π pulses, we consider the case when $\Delta = 0$. Then

$$P_e(t) = |C_e(t)|^2 = \sin^2(\zeta t/2\hbar)$$

$$P_g(t) = \cos^2(\zeta t/2\hbar)$$

Define the atomic inversion as $W(t) = P_e(t) - P_g(t) = -\cos(\zeta t/\hbar)$. Thus we see how the probability of an atom exposed to a resonant field being in one state or the other is dependent upon the time exposed to the field. We make this explicit in the following table.

Pulse→	π: $t = \pi\hbar/\zeta$	$\pi/2$: $t = \pi\hbar/2\zeta$
W(t)	1	0
Status	$C_e = 1$, $C_g = 0$	$C_e = i/\sqrt{2}$, $C_g = 1/\sqrt{2}$

Thus we see the necessity for monokinetic Rydberg atoms (though in actuality one gets quasimonokinetic Rydberg atoms [3]). The speed of the atoms will determine their time exposed to the R_1 and R_2 fields, and thus the state function. In order to obtain the statistics necessary to make quantum mechanical measurements, we desire the ability to nearly perfectly replicate the initial state (the state preparation procedure). The Paris group demonstrated this ability and the Rabi oscillation in 1995 [5] under a similar but simpler apparatus shown in Figure 3.1. They repeat the state preparation on a large set of Rydberg atoms, select those of a certain speed which will result in a corresponding exposure time to the resonant field, and record the the state of the atom after exiting the field. For each selected t, about 20,000 are measured and their state probability ($|e\rangle \rightarrow +1$, $|g\rangle \rightarrow +0$) is averaged.
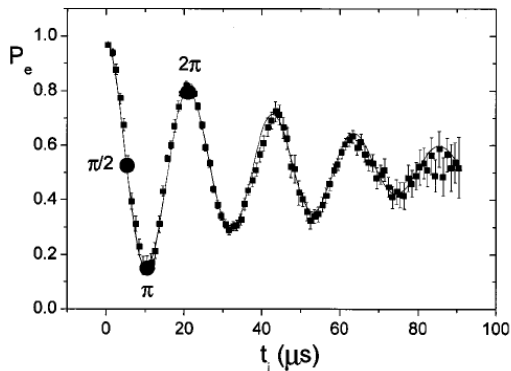

FIGURE 3.2. Experimental Rabi oscillations from [4, 5] Paris group. Initial state is $|e\rangle$ with a π pulse corresponding to a collective shift to the ground state. The dampening of the oscillation is due to technical imperfections in the experiment.

Now let us explore the progress of a Rydberg atom in the larger Paris group experiment. We will follow one atom, and suppose it starts out simply in the excited state, $|\psi_a\rangle = |e\rangle$. The atom has a velocity such that its time through the R1 resonance chamber is a $\pi/2$ pulse, and the atom exits in the state, ideally,

(3.2) $$|\psi_a\rangle = \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle)$$

8

However, realistically we expect the detuning will be non-zero, which introduces a phase difference,

$$|g\rangle \to \frac{|g\rangle + \exp(i\phi)|e\rangle}{\sqrt{2}}, \quad |e\rangle \to \frac{-\exp(-i\phi)|g\rangle + |e\rangle}{\sqrt{2}}$$

Here $\phi \approx \Omega_R T$ where T is the exposure time to the second "Ramsey" beam. We see that the atom has two identical paths for ending up in each of the electronic states, and thus enters the superposition.
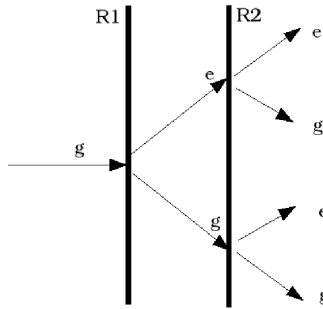

FIGURE 3.3. Possible paths for the wave function.

The probability of finding the atom in $|g\rangle$ is thus the squared sum of the amplitude of the two possible paths, i.e.,

$$(3.3) \qquad\qquad P_g = \frac{1 - \cos\phi}{2}$$

This is exactly what the Paris team found in one such experiment [4]. There result is shown in Figure 3.4.
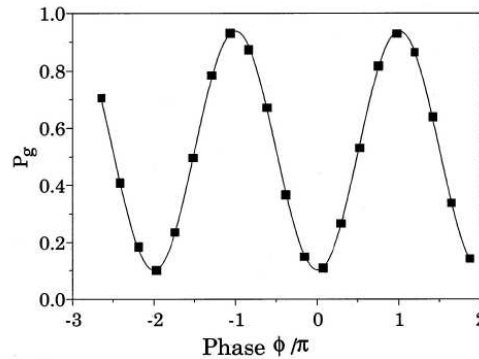

FIGURE 3.4. Experimental evidence of Ramsey fringes from the superposition of two quantum paths (from [4]). The fringes have 85% contrast due to various non-idealities in the experimental set-up.

Thus we have seen both theoretical and real-world examples of quantum effects that must be removed by decoherence.

We now note that the conditions we have thus far considered are highly artificial. What if we consider an interaction with the environment? A simple model for an environmental Hamiltonian might be a set of harmonic oscillators,

$$H_E = \sum_j \hbar\omega_j b_j^\dagger b_j$$

As developed elsewhere, we can write the interaction Hamiltonian as

$$H_I = \sum_j g_j(a^\dagger b_j + ab_j^\dagger)$$

A reasonable way to look at this is to see that if the field gains a photon $(a^\dagger)$ then the single oscillator should loose one b_j and vice versa; the prefactor g_j is a coupling constant that will generally depend on the specifics of the system. The system is now governed by the total Hamiltonian,

$$H = \overbrace{\hbar\omega a^\dagger a + \sum_j \hbar\omega_j b_j^\dagger b_j}^{H_0} + \underbrace{\sum_j g_j(a^\dagger b_j + ab_j^\dagger)}_{H_I}$$

Let us call the new density operator corresponding to our environmentally coupled system ρ_{SE} (S: Single original oscillator; E: Environment) where the individual density operators can be retrieved via a trace, i.e.

$$\rho_S = Tr_E(\rho_{SE}) = \sum_E \langle E|\rho_{SE}|E\rangle \quad \text{Trace over environment states}$$

$$\rho_E = Tr_S(\rho_{SE}) = \sum_S \langle S|\rho_{SE}|S\rangle \quad \text{Trace over local states}$$

The dynamics of the system evolve as (Liouville equation),

$$(4.1) \qquad i\hbar\frac{d\rho_{SE}}{dt} = [H, \rho_{SE}]$$

We commit a unitary transform to simplify this equation as follows (the so-called interaction picture). Let $U_o = \exp(\frac{i}{\hbar}H_0 t)$, then

$$\rho_{se} = U_o\rho_{SE}U_o^\dagger$$

We can then easily find that (since $[H_0, H_0] = 0$, letting $U_oH_IU_o^\dagger \equiv H_i$),

$$\frac{d\rho_{se}}{dt} = \frac{i}{\hbar}[H_0 - U_oHU_o^\dagger, \rho_{se}]$$

$$(4.2) \qquad\qquad = -\frac{i}{\hbar}[H_i, \rho_{se}]$$

With $n = a^\dagger a$, it is demonstrable that,

$$\exp(\eta n)a\exp(-\eta n) = \exp(-\eta)a$$

$$(4.3) \qquad \exp(\eta n)a^\dagger\exp(-\eta n) = \exp(\eta)a^\dagger$$

And so we have,

$$(4.4) \qquad H_i = \sum_j \hbar g_j\left(a^\dagger b_j \exp(i(\omega - \omega_j)t) + ab_j^\dagger \exp(-i(\omega - \omega_j)t)\right)$$

We now follow Orszag and, avoiding some complicated calculations, present a path to the so called master equation[8]. Equation 4.2 is integrated and then reapplied to yield,

$$(4.5) \qquad \frac{d\rho_{se}}{dt} = \frac{1}{i\hbar}[H_i, \rho_{se}(0)] - \frac{1}{\hbar^2}\int_0^t [H_i(t'), [H_i(t'), \rho_{se}(t')]]dt'$$

It is typically not a bad approximation to assume that the environment (bath) is large enough to be unperturbed by the local system, i.e. we assume that (Markovian assumption):

$$\rho_{se}(t) = \rho_s(t) \otimes \rho_e(0)$$

It follows that

$$Tr_B[H_i(t), \rho_{se}(0)] = 0$$

Taking the trace of the Equation 4.5 gives,

(4.6)
$$\frac{d\rho_s}{dt} = -i\Delta\omega[a^\dagger a, \rho_s(t)] + A[a, \rho_s(t)a^\dagger] + A[a\rho_s(t), a^\dagger] + B[a^\dagger, \rho_s(t)a] + B[a^\dagger\rho_s(t), a]$$

Here,

$$\sum_j \quad \to \quad \int D(\omega_j)d\omega_j$$

$$\Delta\omega \quad = \quad P\int_0^\infty \frac{g(\omega_j)^2 D(\omega_j)}{\omega - \omega_j}d\omega_j$$

$$Tr_e(b_j^\dagger b_k \rho_e(0)) \quad = \quad \delta_{jk}\langle n_j \rangle$$

$$Tr_e(b_j b_j \rho_e(0)) \quad = \quad 0$$

$$A \quad = \quad \pi g(\omega)^2 D(\omega)(1 + \langle n(\omega)\rangle)$$

$$B \quad = \quad \pi g(\omega)^2 D(\omega)\langle n(\omega)\rangle$$

Let $\gamma = 2(A - B) = 2\pi g(\omega)^2 D(\omega)$. The master equation becomes

(4.7)
$$\frac{d\rho_s}{dt} = -i\Delta\omega[a^\dagger a, \rho_s(t)] - \frac{\gamma}{2}(1+\langle n(\omega)\rangle)(\rho_s(t)a^\dagger a + a^\dagger a\rho_s(t) - 2a\rho_s(t)a^\dagger) - \frac{\gamma}{2}\langle n(\omega)\rangle(\text{c.c.})$$

We simplify things even further by considering only the simplest of baths, where $T \approx 0$ thus $\langle n(\omega)\rangle \approx 0 \approx \Delta\omega$, and the master equation is simply

$$(4.8) \qquad \frac{d\rho_s}{dt} = -\frac{\gamma}{2}(\rho_s(t)a^\dagger a + a^\dagger a\rho_s(t) - 2a\rho_s(t)a^\dagger)$$

Now we are ready to examine the evolution of our simple model. We recall that it was χ which Decoherence need destroy to produce the classical state of ρ_s. We now apply the master equation to this system and see what happens to these off-diagonal terms.

Conventionally, we start with the so-called normally ordered characteristic function,

$$X_N(\eta, t) = Tr(\rho_s(t)\exp(\eta a^\dagger)\exp(-\eta * a))$$

We take the time derivative of this function, and using the property

$$[a, f(a, a^\dagger)] = \hbar\frac{\partial f(a, a^\dagger)}{\partial a^\dagger}$$

$$[a^\dagger, f(a, a^\dagger)] = -\hbar\frac{\partial f(a, a^\dagger)}{\partial a}$$

$$[a^\dagger a, \exp(\eta a^\dagger)\exp(-\eta^* a)] = \eta a^\dagger)\exp(\eta a^\dagger)\exp(-\eta^* a) + \eta^*\exp(\eta a^\dagger)\exp(-\eta^* a)a$$

11

producing [8, 9]

$$(4.9) \quad \frac{\partial X_N}{\partial t} = Tr\left(\frac{d\rho}{dt}\exp(\eta a^\dagger)\exp(-\eta^* a)\right)$$

$$(4.10) \quad = -\frac{\gamma}{2}\left(\eta\frac{\partial X_N(\eta, t)}{\partial \eta} + \eta^*\frac{\partial X_N(\eta, t)}{\partial \eta*}\right)$$

$$(4.11) \quad = \frac{\partial X_N(\eta, t)}{\partial \eta(t)}\frac{\partial \eta(t)}{\partial t} + \frac{\partial X_N(\eta, t)}{\partial \eta^*(t)}\frac{\partial \eta^*(t)}{\partial t}$$

The solution to this equation is of the form

$$X_N(\eta, t) = X_N\left(\eta\exp\left(\frac{-\gamma t}{2}\right), 0\right)$$

Where $\eta \to \eta(t) = \eta\exp\left(\frac{-\gamma t}{2}\right)$. Finding our initial conditions as

$$\begin{aligned} X_N(\eta, 0) &= Tr(\rho(0)\exp(\eta a^\dagger)\exp(-\eta^* a)) \\ &= \sum_{i,j}\left(\langle\alpha_j|\exp(\eta a^\dagger)\exp(-\eta^* a)|\alpha_i\rangle\right) \\ (4.12) \quad &= \sum_{i,j}\langle\alpha_j|\alpha_i\rangle\exp(\eta\alpha_j^* - \eta^*\alpha_i) \end{aligned}$$

It then follows that,

$$(4.13) \quad X_N(\eta, 0) = \sum_{i,j}\langle\alpha_j|\alpha_i\rangle\exp(\eta\alpha_j^* - \eta^*\alpha_i)\exp(-\frac{\gamma t}{2})$$

Now it is time to note that since

$$|\alpha\rangle = \exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

And thus,

$$\langle\alpha_1|\alpha_2\rangle = \exp(-(|\alpha_1|^2 + |\alpha_2|^2)/2)\sum_{n=0}^{\infty}\frac{\alpha_1^n\alpha_2^n}{\sqrt{n!}}$$

and $\alpha \to \alpha\exp(\gamma t/2)$, the density matrix becomes,

$$(4.14) \quad \rho_{i,j} = \langle\alpha_i|\alpha_j\rangle^{1-\exp(-\gamma t)}|\alpha_j\exp(-\gamma t/2)\rangle\langle\alpha_i\exp(-\gamma t/2)|$$

When $\gamma t \ll 1$ we can approximate the prefactor on the cross terms by $\exp(-2|\alpha|^2\gamma t)$. It is conventional to define $t_c = 1/2\gamma|\alpha|^2$ so that, along with the condition that $\alpha \to \alpha\exp(-\gamma t/2)$, our interference term becomes,

$$(4.15) \quad \langle q'|\rho(t)|q'\rangle = I_1^2(q') + I_2^2(q') + 2I_1(q')I_2(q')\cos\theta(t)\exp(-t/t_c)$$

In terms of the density matrix,

$$\rho_s \quad = \quad \int_{-\infty}^{\infty}\begin{pmatrix} I_1^2 & I_1 I_2\cos\theta(t) \\ I_2 I_1\cos\theta(t) & I_2^2 \end{pmatrix}dq'$$

$$\overset{\text{Decoherence}}{\Longrightarrow} \quad \int_{-\infty}^{\infty}\begin{pmatrix} I_1^2 & I_1 I_2\cos\theta(t)\exp(-t/t_c) \\ I_2 I_1\cos\theta(t)\exp(-t/t_c) & I_2^2 \end{pmatrix}dq'$$

$$\longrightarrow \quad \int_{-\infty}^{\infty}\begin{pmatrix} I_1^2 & 0 \\ 0 & I_2^2 \end{pmatrix}dq'$$

$$\overset{\text{Measurement Problem}}{\Longrightarrow} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The diagonal states will decay into the ground state as they reach equilibrium with the bath. However, this decay will generally be much slower than the decoherence. Finally we must note that decoherence has brought us into a classical density matrix, but the question still remains about which state we will find the particle in when an actual measurement is taken.

We summarize that decoherence is that part of dampening, caused by a coupling to an environment, which produces a decay (a stronger decay) of off-diagonal elements of a density matrix. Thus, in short time, the local quantum density matrix assumes a classical appearance. Since Decoherence is much quicker than normal dissipation, it has become a major engineering problem in the search for a quantum computer.

5. Experimental Example of Decoherence

We now engage the full apparatus seen in Figure 3.1. The cavity (C) is loaded with a coherent field that is off resonance with the $|e\rangle \rightarrow |g\rangle$ transition frequency but instead is tuned close to a seperate transition ($|e\rangle \rightarrow |i\rangle$ transition (n=52, 48.180 GHz)). The cavity is designed so that the atom traverses the field adiabatically [3]. An atom in the state $|e\rangle$ will thus interact with this field, and so we need to consider the full quantum case of Rabi oscillation as the phase of the coherent cavity field will be shifted. However, the principle is the same as in the semi-classical case, so that we simply state the result. The coherent field will be phase shifted by the presence of any atom in the $|e\rangle$ state.

After leaving R1, the atom is in the state

$$(5.1) \qquad |\psi_1\rangle = \frac{|e\rangle + |g\rangle}{\sqrt{2}}$$

It passes through (C) adiabatically so that no exchange of photons is allowed. However, the coherent field $|\alpha\rangle$ in the chamber will be phased shifted if the atom is in state $|e\rangle$, and as before, a proper selection of experimental variables and atomic velocity ($\approx 100m/s$ [3]) can bring this shift into a particular value (π in this case). More generally (which we will follow in the dissipation case) the phase shift is $\exp(-i\pi a^\dagger a)$.

Thus the field and atom become entangled after passage, i.e.,

$$(5.2) \qquad |\psi_{1,f}\rangle = \frac{|e; -\alpha\rangle + |g; \alpha\rangle}{\sqrt{2}}$$

As before, the atom now passes through R2 where it undergoes an $\pi/2$ pulse such that the state becomes,

$$(5.3) \qquad |\psi_{1,f}\rangle = \frac{|e; -\alpha\rangle - |e; \alpha\rangle + |g; \alpha\rangle + |g; -\alpha\rangle}{2}$$

The field state is now entangled (in a coherent superposition) so that a measurement of the atom (finding it in either $|e\rangle$ or $|g\rangle$) will "project" the field into the state,

$$|\psi_f\rangle = \frac{|\alpha\rangle + \exp(i\mu)|-\alpha\rangle}{\sqrt{2(1 + \cos\mu \exp(-2a^2))}} \quad \begin{cases} \mu = 0 & \text{Atom found in } |g\rangle \\ \mu = \pi & \text{Atom found in } |e\rangle \end{cases}$$

Next we send in a second atom. We assume that this second atom is sent in quickly enough (a time T later) after the first that relaxation can be neglected. After the second atom passes R1, the system state is given as (taking the normalization factor as $\sqrt{2}$ for simplicity),

$$|\psi_{2,f}\rangle = \frac{1}{2}\left((|e\rangle + |g\rangle) \otimes (|\alpha\rangle + e^{i\mu}|-\alpha\rangle)\right)$$

13

The density operator with decay and decoherence will be,

$$(5.4) \qquad \rho_{af}(T) = \frac{1}{2}(|e\rangle + |g\rangle)(\langle e| + \langle g|) \otimes \rho_f(T)$$

Here $\rho_f(T)$ is the density operator of the field which has undergone the decay and decoherence introduced in the previous section, i.e.

$$
\begin{aligned}
\rho_f(T) =\ & \frac{1}{N^2}(|\alpha e^{-\gamma T/2}\rangle\langle\alpha e^{-\gamma T/2}| + |-\alpha e^{-\gamma T/2}\rangle\langle-\alpha e^{-\gamma T/2}| \\
& + \exp\left(-2|\alpha|^2(1 - e^{-\gamma T})\right)\left(e^{i\mu}| - \alpha e^{-\gamma T/2}\rangle\langle\alpha e^{-\gamma T/2}| \right. \\
& \left. + e^{-i\mu}|\alpha e^{-\gamma T/2}\rangle\langle-\alpha e^{-\gamma t/2}|\right))
\end{aligned}
$$

In cavity (C), the atom becomes entangled with the field so that we can write in the non-decay case,

$$(5.5) \qquad |\psi_{2,f}\rangle = \frac{1}{2}\left(|e;-\alpha\rangle + |g;\alpha\rangle + e^{i\mu}(|e;\alpha\rangle + |g;-\alpha\rangle)\right)$$

The decayed density matrix is,

$$\rho_{af} = \frac{1}{2}\left(|e\rangle\langle e|e^{-i\pi a^\dagger a}\rho_f(T)e^{i\pi a^\dagger a} + |g\rangle\langle g|\rho_f(T) + |e\rangle\langle g|e^{-i\pi a^\dagger a}\rho_f(T) + |f\rangle\langle e|\rho_f(T)e^{i\pi a^\dagger a}\right)$$

Next the atom passes through R2 and the wave function becomes,

$$(5.6) \qquad |\psi_{2,f}\rangle = \frac{1}{2\sqrt{2}}(|e;-\alpha\rangle + |g;-\alpha\rangle + |g;\alpha\rangle - |e;\alpha\rangle$$

$$(5.7) \qquad\qquad + e^{i\mu}(|e;\alpha\rangle + |g;\alpha\rangle + |g;-\alpha\rangle - |e;-\alpha\rangle)$$

With decay, the density matrix is,

$$
\begin{aligned}
\rho_{af} =\ & \frac{1}{4}((|e\rangle + |g\rangle)(\langle e| + \langle g|)e^{-i\pi a^\dagger a}\rho_f(T)e^{i\pi a^\dagger a} + (-|e\rangle + |g\rangle)(-\langle e| + \langle g|)\rho_f(T) \\
& (|e\rangle + |g\rangle)(-\langle e| + \langle g|)e^{-i\pi a^\dagger a}\rho_f(T) + (-|e\rangle + |g\rangle)(\langle e| + \langle g|)\rho_f(T)e^{i\pi a^\dagger a})
\end{aligned}
$$

Following Davidovich et al., we write the probability of finding the second atom in $|e\rangle$ or $|g\rangle$ as

$$
\begin{aligned}
P\left(\begin{smallmatrix} g \\ e \end{smallmatrix}\right)(T) =\ & \frac{1}{2}\left(1 \pm \mathrm{Re}(Tr(e^{-i\pi a^\dagger a}\rho_f(T)))\right) \\
=\ & \frac{1}{2}(1 \pm \mathrm{Re}(\frac{1}{N^2}(\langle\alpha e^{-\gamma T/2}| - \alpha e^{-\gamma T/2}\rangle + \langle-\alpha e^{-\gamma T/2}|\alpha e^{-\gamma T/2}\rangle \\
& + \exp(-2|\alpha|^2(1 - e^{-\gamma T})(e^{i\mu}\langle\alpha e^{-\gamma T/2}|\alpha e^{-\gamma T/2}\rangle + e^{-i\mu}\langle-\alpha e^{-\gamma T/2}| - \alpha e^{-\gamma T/2}\rangle))) \\
=\ & \frac{1}{2}(1 \pm \mathrm{Re}\left(\frac{2}{N^2}\left(\exp(-2|\alpha|^2 e^{-\gamma T}) + \exp(-2|\alpha|^2(1 - e^{-\gamma T}))\cos\mu\right)\right)) \\
=\ & \frac{1}{2}\left(1 \pm \frac{\exp(-2|\alpha|^2 e^{-\gamma T}) + \exp(-2|\alpha|^2(1 - e^{-\gamma T}))\cos\mu}{1 + (\cos\mu)e^{-2|\alpha|^2}}\right)
\end{aligned}
$$

Since $\mu = 0$ corresponds to the first atom being detected in the ground state, and $\mu = \pi$ corresponds to the first atom being detected in the excited state, one can calculate $P(g,e;T)$, $P(e,e;T)$, and so on. Davidovich et al., for example, plot P(g,e,T) and P(e,e,T) as shown in Figure 5.1.

An experiment testing these predictions was first carried out in 1996 by the Paris team [1]. Based on these theoretical predictions, this team measured the average conditional probability, i.e.

$$(5.8) \qquad \eta = \left(\frac{P_{ee}}{P_{ee} + P_{eg}}\right) - \left(\frac{P_{ge}}{P_{ge} + P_{gg}}\right)$$

They find strong agreement with theoretical predictions, as seen in Figure 5.2.
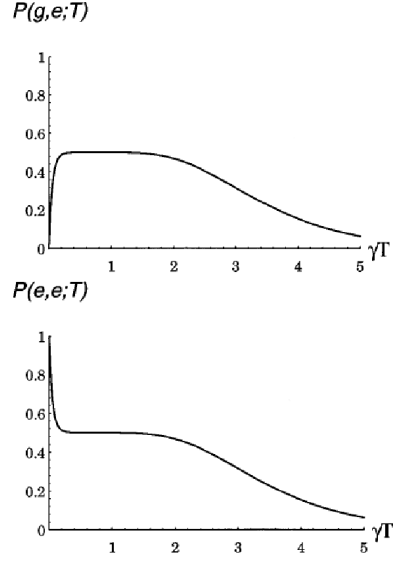
P(g,e;T)

P(e,e;T)

FIGURE 5.1. Probabilities of detecting second atom in state $|e\rangle$ depending on initial state of the first atom (numerical from [3]). Initial decay is decoherence effect. Plateau at 0.5 is classical like-state. Decay at large T corresponds to relaxation of system into the ground state ($P_{gg} \to 1$ as $t/\gamma T \to \gg 1$.)
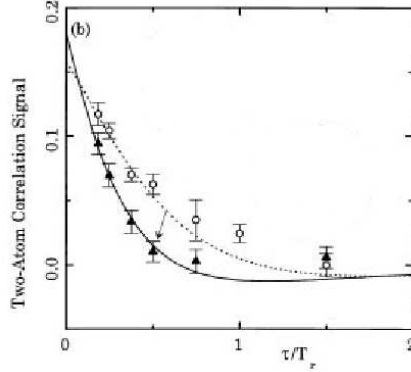

FIGURE 5.2. Correlation signal decay due to decoherence. The top line and bottom line represents different phases induced by different detuning in the (C) chamber. The dashed line corresponds to a higher detuning and thus a less intefered state and decoheres slower accordingly. Circles and triangles correspond to experimental results. $\tau = 40\mu s$ and T is varied. See [1] for details.

6. CONCLUSION

The Paris group, and other research groups we will discuss in our presentation, have demonstrated the action of decoherence in reducing quantum density matrices into classical form via a coupling of the system under study with a background environment.

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