A BRIEF INTRODUCTION TO THE LORENZ GAUGE & THE QUANTIZATION OF THE ELECTRIC FIELD

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Abstract. What is too often referred to as the Lorentz (it was Ludwig Lorenz in 1867, not Hendrik Lorentz, who first proposed this gauge) is commonly used and so we present a brief introduction.

1. Potential Equations ↔ Maxwell's Equation.

In Lorenz's paper [1] he begins with scaler and vector potentials (in retarded from) and derives Maxwell's equations from these equations. Typically, texts start with Maxwell's equations and develop the Lorenz Gauge [2, 3] which has the benefit of seeming less ad hoc. Here we present a graphical representation of the development.

We begin with the Maxwell equations in general form,

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0 \qquad \nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

We note that

$$\nabla \cdot B = 0 \;\Rightarrow\; B \equiv \nabla \times A \ni \nabla \times E = -\frac{\partial}{\partial t}(\nabla \times A)$$

$$\therefore \nabla \times \left(E + \frac{\partial A}{\partial t} \right) = 0 \;\Rightarrow\; E + \frac{\partial A}{\partial t} = -\nabla V \;\ni\; E = -\nabla V - \frac{\partial A}{\partial t}$$

$$\ni\; \nabla \cdot E = \frac{\rho}{\epsilon_0} \;\rightarrow$$

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot A) = -\frac{\rho}{\epsilon_0} \tag{1.1}$$

Finally we note that

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \;\rightarrow\; \nabla \times (\nabla \times A) = \mu_0 J - \mu_0 \epsilon_0 \nabla \left(\frac{\partial V}{\partial t} \right) - \mu_0 \epsilon_0 \frac{\partial^2 A}{\partial t^2}$$

Since

$$\nabla \times (\nabla \times A) = \partial_j \epsilon_{ijk}(\partial_i A_j) + \partial_j \epsilon_{ijk}(\partial_j A_i) = \nabla(\nabla \cdot A) - \nabla^2 A,$$

we have

$$\left(\nabla^2 A - \mu_0 \epsilon_0 \frac{\partial^2 A}{\partial t^2} \right) - \nabla \left(\nabla \cdot A + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) = -\mu_0 J \tag{1.2}$$

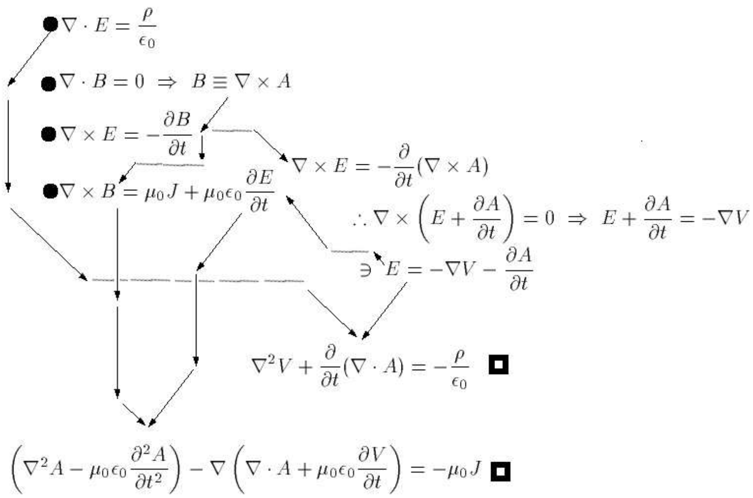

FIGURE 1.1. Maxwell's Equations ↔ Potential Equations

2. Gauges

Equations 1.1,1.2 are not unique. Suppose $A' = A + a$ and $V' = V + b$, we do not violate Maxwell's equations if

$$\nabla \times a = 0 \;\Rightarrow a \;= \nabla \lambda$$

and

$$\nabla b + \frac{\partial a}{\partial t} = 0 \;\ni \nabla \left(b + \frac{\partial \lambda}{\partial t}\right) = 0 \;\Rightarrow\; b + \frac{\partial \lambda}{\partial t} = f(t),\; \nabla f = 0.$$

Typically one redefines $\lambda \rightarrow \lambda + \int_0^t f(t')dt'$, returning the general gauge transform,

$$A' = A + \nabla \lambda \tag{2.1}$$

$$V' = V - \frac{\partial \lambda}{\partial t} \tag{2.2}$$

Two common gauges are the Coulomb and Lorenz. The Coulomb gauge has us take λ so that $\nabla \cdot A = 0$. Thus Equation 1.1 simplifies to a harmonic equation:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \;\ni\; V(r,t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r',t)}{|r - r'|} d\tau'$$

Unfortunately this does not do much to simplify the defining equation for the vector potential,

$$\nabla^2 A - \mu_0 \epsilon_0 \frac{\partial^2 A}{\partial t^2} = -\mu_0 J + \mu_0 \epsilon_0 \nabla \left(\frac{\partial V}{\partial t}\right)$$

The Lorenz gauge has us chose $\nabla \cdot A + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} = 0$. Define the d'Alembertian as $\nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \equiv \Box$. Then the Lorenz gauge reduces the potential equations to

$$\Box^2 V = -\frac{\rho}{\epsilon_0} \tag{2.3}$$

$$\Box^2 A = -\mu_0 J \tag{2.4}$$

3. Quantization of an electromagnetic field with standing waves

Maxwell's equations for a source free environment are

$$\nabla \cdot \mathbf{D} = \nabla \cdot \mathbf{B} = 0 \tag{3.1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3.2}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \tag{3.3}$$

In this environment, $B = \mu_o H$ and $D = \epsilon_o E$

The simplicity of these four equations begs for even further simplification, whereby we introduce the vector potential,

$$\nabla \times \mathbf{A} \equiv \mathbf{B} \quad \longrightarrow \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V$$

There are many ways to fit these equations while maintaining the validity of the Maxwell equations. The coulomb potential suffices and is preferrable due to its simplicity: $\nabla \cdot \mathbf{A} = \mathbf{0}$, $V = 0$. The identity $\nabla \times \nabla \times A = \nabla(\nabla \cdot A) + (\nabla \cdot \nabla)A$ coupled with our previous assertation and equation 3 above yield

$$\nabla^2 A = \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} \tag{3.4}$$

We will quantize this centralized magnetic potential. To completely specify the field we would have to describe its values for all points in space; it is customary to develop the quantization in a theoretical cube, and then let the volume of the cube expand to infinity to accomplish a full discription.

Our derivation can consider standing or plane waves. The case of standing waves is quicker. We assume the magnetic potential has a solution of form

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{\epsilon_o}\sum_l q_l \mathbf{u}_l(\mathbf{r}) \tag{3.5}$$

Under this seperation of variables,

$$q_l \nabla^2 u_l = \frac{u_l}{c^2}\frac{d^2 q_l}{dt^2} \quad \longrightarrow \quad \frac{c^2}{u_l}\nabla^2 u_l = \frac{1}{q_l}\frac{d^2 q_l}{dt^2} \equiv -\omega_l^2 \tag{3.6}$$

$$\ni \ \nabla^2 u_l + \frac{\omega_l^2}{c^2} u_l = 0, \quad \frac{d^2 q_l}{dt^2} + \omega_l^2 q_l = 0 \tag{3.7}$$

Being in the standing wave regime, there can be no currents on the boundary, implying that $u_l|_{tangential} = 0$ and $\nabla \times u_l|_{normal} = 0$. But of course, we have already doomed ourselves to the fact that $\nabla \cdot u_l = 0$. These consequences become important as follows.

The energy stored by our electromagnetic field is

$$H = \frac{1}{2}\int (\epsilon_o E^2 + \mu_o H^2) dv = \frac{1}{2}\int \left(\epsilon_o \left(\frac{\partial A}{\partial t}\right)^2 + \mu_o \left(\nabla \times A\right)^2\right) dv$$

We assume via our target solution that our spatial modes will have the orthogonality

$$\int u_l \cdot u_m dv = \delta_{l,m} \tag{3.8}$$

Using this orthogonality, we have

$$H = \frac{1}{2}\sum_l \left(\frac{dq_l}{dt}\right)^2 \int u_l^2 dv + \frac{c^2}{2}\sum_l q_l^2 \int (\nabla \times u_l)^2 \, dv \tag{3.8}$$

The rightmost integral can be taken with an algebraic manipulation. We examine it as follows:

$$(\nabla \times u_l) \cdot (\nabla \times u_l) \equiv B \cdot (\nabla \times A)$$

A snap shot of the latter yields:

$$B_k(\nabla_i A_j) = \nabla_i(A_j B_k)$$

This suggest the generating equation

$$\nabla \cdot (A \times B) \ : \ \nabla_i A_j B_k - \nabla_i A_k B_j = B_k(\nabla_i A_j) - A_k(\nabla_i B_j)$$

Thus our vector identity is

$$\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$$

More appropriate for our needs, we rearrange the previous equation to find:

$$B \cdot (\nabla \times A) = \nabla \cdot (A \times B) + A \cdot (\nabla \times B)$$

Reidentifying $A = u_l$ and $B = \nabla \times u_l$, we have

$$(\nabla \times u_l) \cdot (\nabla \times u_l) = \nabla \cdot (u_l \times \nabla \times u_l) + u_l \cdot (\nabla \times \nabla \times u_l)$$

Remembering that this is under a volume integral, we quickly see that the first term Stokes away as a surface integral which our previously established boundary conditions founded as zero. The second term is taken care of by the use of a previous vector identity. Catching up, we have

$$H = \frac{1}{2}\sum_l \left(\frac{dq_l}{dt}\right)^2 + \frac{c^2}{2}\sum_l q_l^2 \int u_l \cdot (\nabla(\nabla \cdot u_l) - \nabla^2 u_l) dv \tag{3.8}$$

With equation 7 and our boundary conditions, this becomes,

$$H = \frac{1}{2}\sum_l \left(\left(\frac{dq_l}{dt}\right)^2 + \omega_l^2 q_l^2\right) \equiv \sum_l H_l \tag{3.8}$$

This form is fully equivalant to the harmonic oscillator, and with the following set:

$$p_l = \frac{dq_l}{dt} \tag{3.9}$$

$$q_l = \sqrt{\frac{\hbar}{2\omega_l}}(a_l^\dagger + a_l) \tag{3.10}$$

$$p_l = i\sqrt{\frac{\hbar\omega_l}{2}}(a_l^\dagger - a_l) \tag{3.11}$$

we complete the standing wave quantization by concluding that

$$H = \frac{1}{2}\sum_l \hbar\omega_l \left(a_l^\dagger a_l + a_l a_l^\dagger\right) \tag{3.11}$$

4. Quantization of an electromagnetic field using plane waves

Also common to find in the literature is the following version of the quantization of the electromagnetic field.

We rejoin the previous discussion, interjecting now the more complicated attempted solution.

$$\mathbf{A}(\mathbf{r},t) = \sum_m \sqrt{\frac{\hbar}{2\omega_m \epsilon_o}} \left(a_m(t)\mathbf{u}_m(\mathbf{r}) + a_m^\dagger(t)\mathbf{u}_m^*(\mathbf{r})\right) \tag{4.0}$$

As before, it follows that,

$$\nabla^2 u_m(r) + \frac{\omega^2}{c^2} u_m(r) = 0 \tag{4.1}$$

$$\frac{\partial^2 a_m}{\partial t^2} + \omega^2 a_m = 0 \tag{4.2}$$

It follows, using plane waves,

$$a_m(t) = a_m e^{-i\omega_m t} \tag{4.3}$$

$$a_m^\dagger(t) = a_m^\dagger e^{i\omega_m t} \tag{4.4}$$

$$u_m(r) = \frac{e_m}{\sqrt{v}} e^{ik_m \cdot r} \quad @ \quad k_m^2 = \frac{\omega_m^2}{c^2} \tag{4.5}$$

Here, following the notation of Orszag, e_m represents the appropriate unit vector. Requiring only periodic boundary conditions: $A(\mathbf{r} + L\mathbf{e_m}) = A(\mathbf{r}) \ni \mathbf{k_m} = 2\pi(m_1\mathbf{i} + m_2\mathbf{j} + m_3\mathbf{k})/L$. We thus have,

$$\mathbf{A}(\mathbf{r},t) = \sum_m \sqrt{\frac{\hbar}{2\omega_m \epsilon_o v}} \mathbf{e_m} \left(a_m e^{i(k_m \cdot r - \omega_m t)} + a_m^\dagger e^{-i(k_m \cdot r - \omega_m t)}\right) \tag{4.5}$$

Again,

$$H = \frac{1}{2} \int \left(\epsilon_o \left(\frac{\partial A}{\partial t}\right)^2 + \frac{1}{\mu_o} \left(\nabla \times A\right)^2\right) dv$$

By sight we see that the result of this integral will come down to:

$$\left(-iae^{\alpha} + ia^\dagger e^{-\alpha}\right) \cdot \left(-iae^{\alpha} + ia^\dagger e^{-\alpha}\right) + \left(iae^{\alpha} - ia^\dagger e^{-\alpha}\right) \cdot \left(iae^{\alpha} - ia^\dagger e^{-\alpha}\right)$$

Thus we are really concerned with:

$$2\left(-iae^{\alpha} + ia^\dagger e^{-\alpha}\right) \cdot \left(-iae^{\alpha} + ia^\dagger e^{-\alpha}\right) = -2aae^{2\alpha} - a^\dagger a^\dagger e^{-2\alpha} + 2\left(aa^\dagger + a^\dagger a\right)$$

Only the latter term survives through the integration (due to the periodic boundary conditions), and we again conclude that:

$$H = \frac{1}{2} \sum_l \hbar\omega_l \left(a_l^\dagger a_l + a_l a_l^\dagger\right) \tag{4.5}$$

References

[1] Lorenz L., *On the Identity of the Vibrations of Light with Electrical Currents*, Philos. Mag. 34, 287-301, 1867

[2] Jackson D., *Classical Electrodynamics*, John Wiley & Sons, 1962

[3] Griffiths D., *Introduction to Electrodynamics*, Prentice Hall, 1981