Intellectual Hedonism

Irrelevant Topics in Physics V

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Random Matrix Theory

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- Random Matrix Theory - Ensemble only: $A_{ij} \in \mathbf{GUE}$

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- The ensemble from which they are chosen from should have the same statistics, thus 'average' predictions should be correct.
- Choose an ensemble of matrices that have the same symmetries as your system.

Ensembles of matrices?

- GOE (Gaussian orthogonal ensemble) probability density:

$$\exp\left(-\frac{N\mathbf{Tr}(H^2)}{\gamma^2}\right)\Pi dH_{\mu\nu}$$

$\Pi dH_{\mu\nu}$ product of differentials of the independent matrix elements, N matrix size, Gaussian factor introduced to render integrals over space convergent (cutoff). Characterized by a single parameter γ, with dimensions of energy. γ Determines the mean-level spacing.

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- Stay with me, pictures are coming!

Typical λ Spacings for different systems

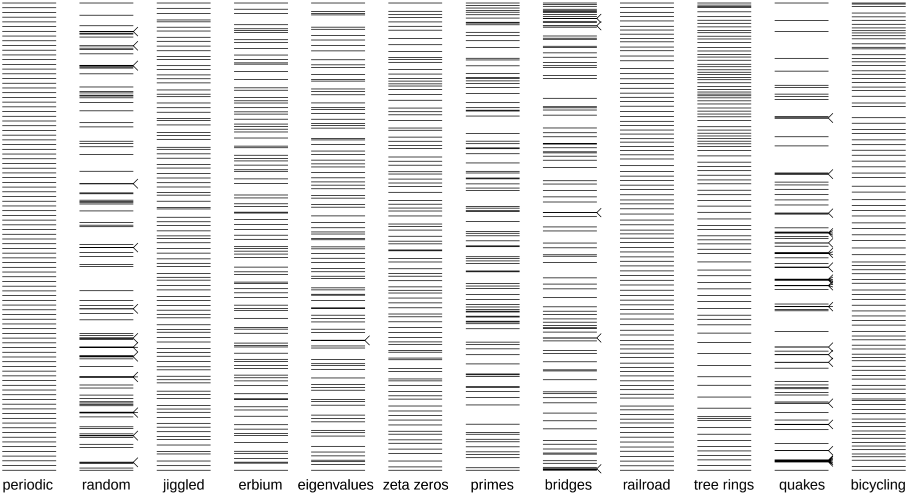


Figure 1. One-dimensional distributions each consist of 100 levels. From left to right the spectra are: a periodic array of evenly spaced lines; a random sequence; a periodic array perturbed by a slight random "jiggling" of each level; energy states of the erbium-166 nucleus, all having the same spin and parity quantum numbers; the central 100 eigenvalues of a 300-by-300 random symmetric matrix; positions of zeros of the Riemann zeta function lying just above the 10^{22}nd zero; 100 consecutive prime numbers beginning with 103,613; locations of the 100 northernmost overpasses and underpasses along Interstate 85; positions of crossties on a railroad siding; locations of growth rings from 1884 through 1983 in a fir tree on Mount Saint Helens, Washington; dates of California earthquakes with a magnitude of 5.0 or greater, 1969 to 2001; lengths of 100 consecutive bike rides.

Quantum Chaos

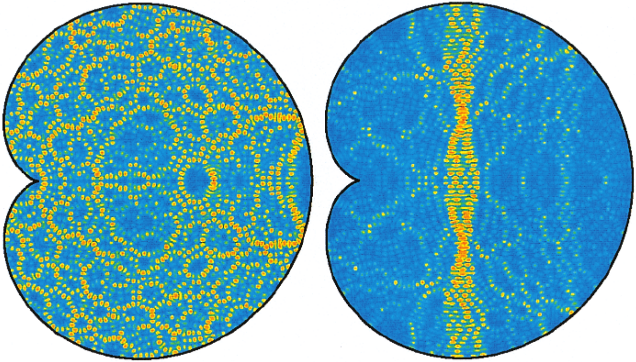


Fig. 1. Probability density of the 1,816th and 1,817th odd eigenstate of a quantum particle trapped in a chaotic heart-shaped region with Dirichlet boundary conditions. The probability of finding the particle at a given point is low in blue regions and high in red regions.

Quantum Chaos as a function of Integrability

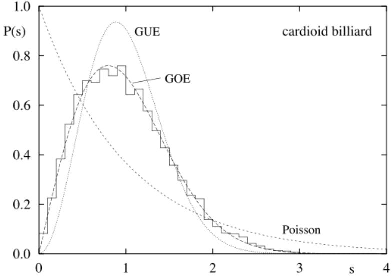


Fig. 2. Level spacing distribution for the energy spectrum of a quantum particle in the chaotic heart-shaped region of Fig. 1 vs. the level spacing distribution for Gaussian Unitary Ensemble, Gaussian Orthogonal Ensemble, and Poisson, respectively.

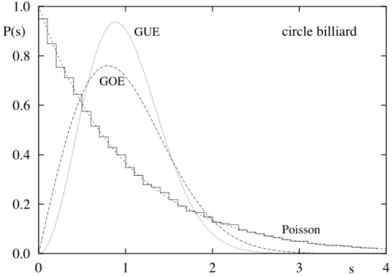
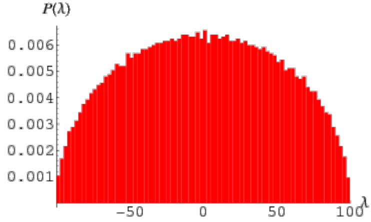
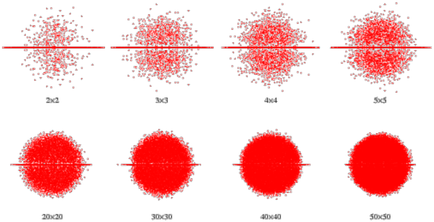


Fig. 3. Level spacing distribution for the energy spectrum of a quantum particle in a circular region vs. the level spacing distribution for Gaussian Unitary Ensemble, Gaussian Orthogonal Ensemble, and Poisson, respectively.

Wigner and Girkos Law

Eigenvalue spacing for Real (Symmetric) Matrix Standard Normal Distributions





Girko's Law predicts eigenvalues spacing will cover the unit disc uniformly.

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$$1 - \left(\frac{\sin \pi x}{\pi x} \right)^2$$

Bonus! Connections to Riemann-Zeta function

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Freeman Dyson walks over and recognizes this as the *exact* same result he got, for the Gaussian Unitary Ensemble!

Complex Temperatures

Since Stat. Mech. was to easy

Motivation comes from the theory of phase transitions:

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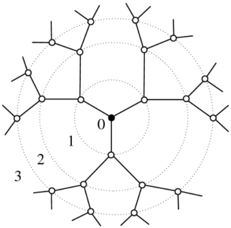
- This can NOT happen in finite systems!
- Can use renormalization, and finite-size scaling tricks to find the critical points

Cayley Trees == Beethe Lattice

As a sample system, look at the Ising Bethe (yes that one) lattice:



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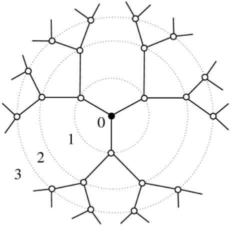
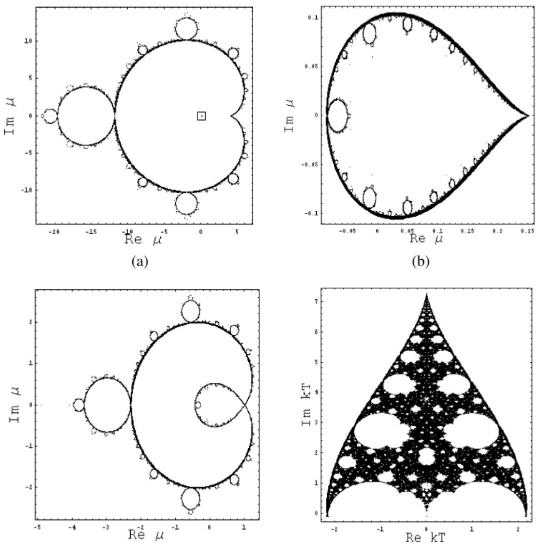
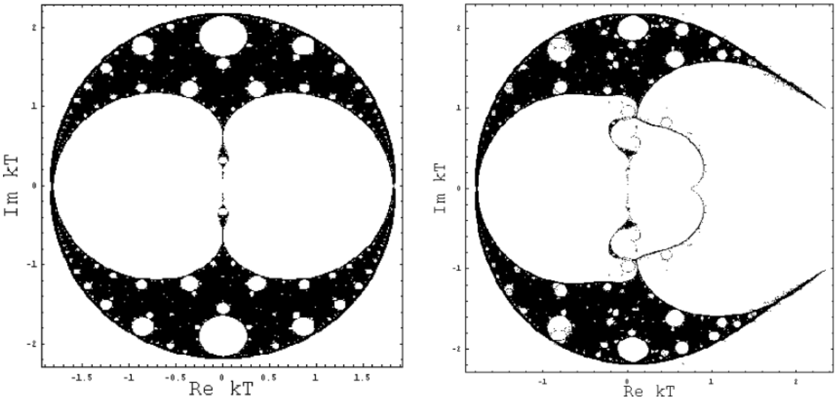


- Often times this model is exactly solvable for a given $\mathcal{H}$.
- Surface area $\propto$ Number of nodes (very unusual!)

Yang-Lee partition function zeros for the Ising Cayley tree



Fisher partition function zeros for the Ising Cayley tree



Partition function zeros for one-dimensional Blume-Capel

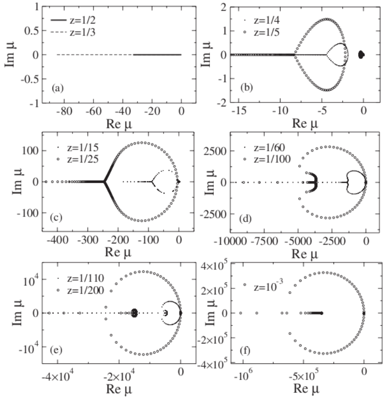


Figure: Yang-Lee Zeros

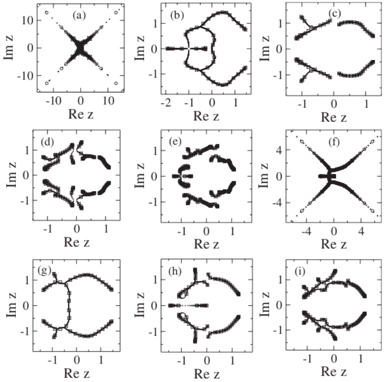


Figure: Fisher Zeros

Stochastic Resonance

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- Applicable to Schmitt riggers, ring-laser experiments, neurological inputs, Josephson Junctions and more...

Simplest example of Stochastic Resonance

Overdamped Brownian motion in bistable potential with periodic forcing:

$$\dot{x}(t) = ((1/2)x^2 - (1/4)x^4) + A_0 \cos(\Omega t + \phi) + \xi(t)$$

$$< \xi(t)\xi(0) >= 2D\delta(t) \quad < \xi(t) >= 0$$

$\xi(t)$ is a Wigner process, ie. white, Gaussian noise. Function has two peaks at $+/-x_m = 1$. In absence of forcing $x(t)$ fluctuates around local minima according to Kramers rate:

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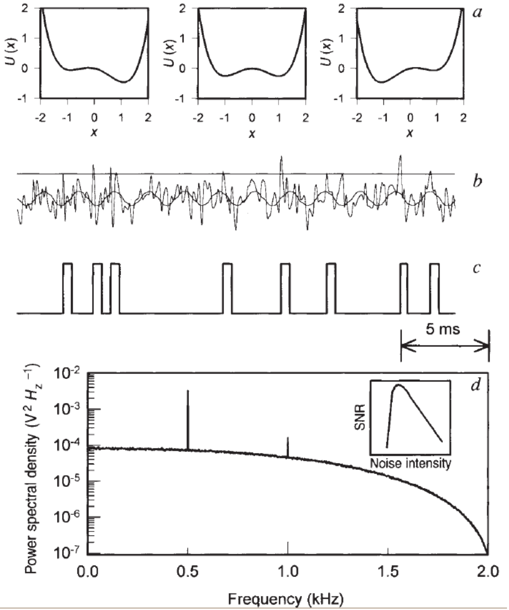
At resonant values of D the 'signal' (ie. the value that Ω can be detected from the noise) is at maximized:

$$SNR \propto \left(\frac{\epsilon \Delta V}{D}\right)^2 e^{-\Delta V/D}$$

SR Potential Example



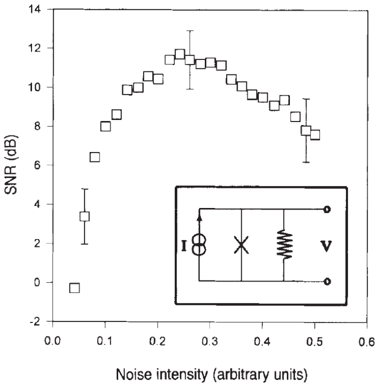


FIG. 4 Main figure, data from a numerical simulation of a trigger-reset system based on a Josephson junction as nonlinear element[48]. Inset, circuit diagram of the Josephson system, consisting of an ideal junction (cross), quasiparticle resistance, and current source *I* which is the sum of three components—constant bias, weak periodic signal and noise. The output is the voltage, *V*.