PHYS 305 - Assignment #6

Due: Friday, February 20^{th}

Make sure your name is listed as a comment at the beginning of all your work.

Purpose: Develop a physical intuition for Fixed Points & Limit Cycles.

The Lotka-Volterra Model

The Lotka-Volterra model is defined by the ODEs

$$\frac{dx(t)}{dt} = x(t)\left(A - By(t)\right)$$

$$\frac{dy(t)}{dt} = -y(t)\left(C - Dx(t)\right)$$

Understand the ODE solver code we developped in class (found on the web page) to solve this model.

- Draw a phase space portrait for this model, pointing out the two *Fixed Points*
- *Guess* the type of *Fixed Points* these are according to the description found in **Nonlinear Dynamics - A Two Way Trip from Physics to Math** by *H.G. Solari, M.A Natiello and G.B. Mindlin* (in-class handout).

A Population Dynamics Model - Fixed Points & Limit Cycle

Another model used in population dynamics, as well as to describe competition between firms in the business world, is described by the following Ordinary Differential equations

$$\frac{du1(t)}{dt} = u1(t)(1 - u2(t)u2(t)) - u2(t)$$

$$\frac{du2(t)}{dt} = u1(t) - u1(t)u2(t)$$

The functions to solve for are $u1(t)$ and $u2(t)$ given their initial values $u1(0)$ and $u2(0)$ at time $t = 0$. This model is a modified version of the *Lotka-Volterra Two Species Model*, also called the *Predator Prey Model*, or the *Resource Consumer Model.*

- Understand in great details the ODE solver code we developped in class (found on the web page) to solve this model. Use it as a starting point to answer the questions below (make minimal changes to it) together with *feeder codes.*

- Calculate and plot the trajectories corresponding to the initial conditions $(u1(0), u2(0)) = (0.1, 0.1)$ and $(0.0, -0.5)$

These trajectories converge toward a closed orbit. This trajectory is called a **Limit Cycle**.

- Choose initial conditions ($u1(0)$,$u2(0)$) to generate this *Limit Cycle* by itself with no transient leading to it. Graph it. Comment on the initial conditions you chose.

This system admits one **Fixed Point**.

- Find the coordinates ($u1$,$u2$) of this *Fixed Point*
- Perform a numerical experiment to decide on the stability of this *Fixed Point*. Comment on your approach and results. *Guess* the type of *Fixed Point* this is according to **Nonlinear Dynamics - A Two Way Trip from Physics to Math** by *H.G. Solari, M.A Natiello and G.B. Mindlin*

Basin of Attraction

Some initial conditions lead to the *Limit Cycle* you have identified above. Others produce orbits that escape to large distance. The area in the ($u1$,$u2$) phase space containing initial conditions that converge to the *Limit Cycle* form the **Basin of Attraction** of this orbit. You are now to draw this *Basin of Attraction*. Here is how to do this:

- Modify your ODE solver code to *break* from the time loop if a trajectory ventures too far from the origin; in practice if the trajectory goes outside of a rectangle $-6 < u1(t) < 3.0$ or $-4.5 < u2(t) < 1.53.0$ is a suffcient criteria to declare the trajectory gone.
- Write a small program, *initial.c*, to generate initial conditions on an equally spaced rectangular grid in $u1, u2$ space to feed initial conditions via *piping* in your ODE solver code. Use the rectangle defined above as the domain of initial conditions to explore.
- Modify further your ODE solver code to output in *stderr* the initial conditions that produced a *bound* trajectory according to the criteria above.
- Produce a graph showing the *Basin of Attraction* on a fine initial conditions grid (step 0.1 in both directions) with the *Limit Cycle* superimposed on it.
- Produce a graph (work of art!) showing the *Basin of Attraction* on a fine initial conditions grid (step 0.1 in both directions) with trajectories initiating from a coarser grid of initial conditions (step 0.5 - in and out of the *Basin of Attraction*) layared on it.
- Comment on the transient behavior of the orbits above. Namely which way do the trajectories that initiate from within the *Basin of Attraction* follow in reaching the *Limit Cycle*. In other words what is the direction of the transient flow leading to the *Limit Cycle*? You may want to plot few trajectories coming from characteristic points of the *Basin of Attraction* to illustrate your answer.