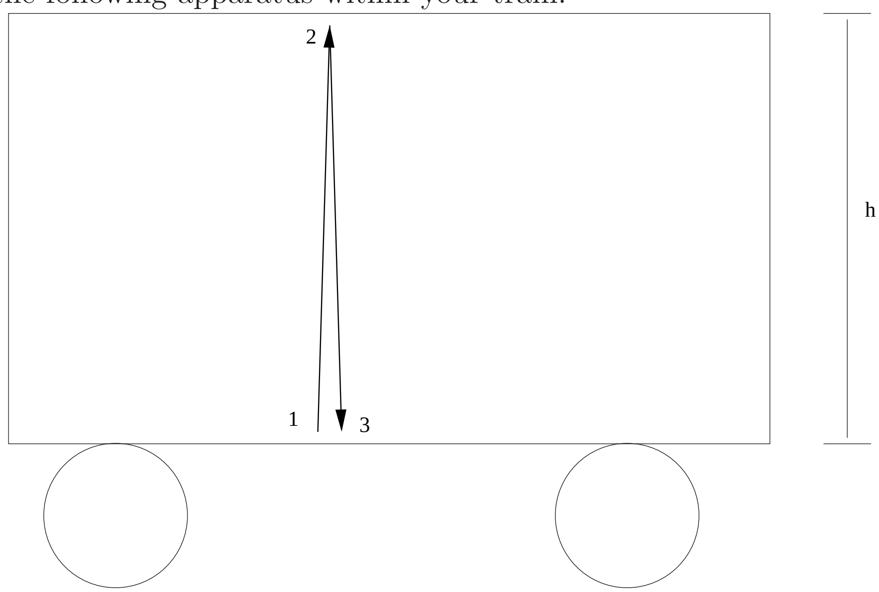
A Simple Experiment with Light

As I've mentioned before, you are in the primed frame (the one inside the train). This is a very special train, one set up with mirrors and lasers, and sophisticated chronometers and the like.

Light (in the form of lasers) is particularly important because of an experiment done in 1887 by Michelson and Morely, which showed that light travels at a constant speed for all observers. Using that fact, and the fact that at low speeds our picture of the universe should be the same as that suggested by Galileo, will be our basis of **special relativity**.

So, consider the following apparatus within your train.



In this "experiment" there are 3 events:

1) A pulse is fired (at the speed of light) from the bottom of the train.

2) The pulse is reflected from the top of the train.

3) The pulse is detected at the bottom of the train again.

View from the primed frame:

Despite the small offset in the picture, events "1" and "3" take place at the same position (according to you), and thus:

$$\Delta x' = 0$$

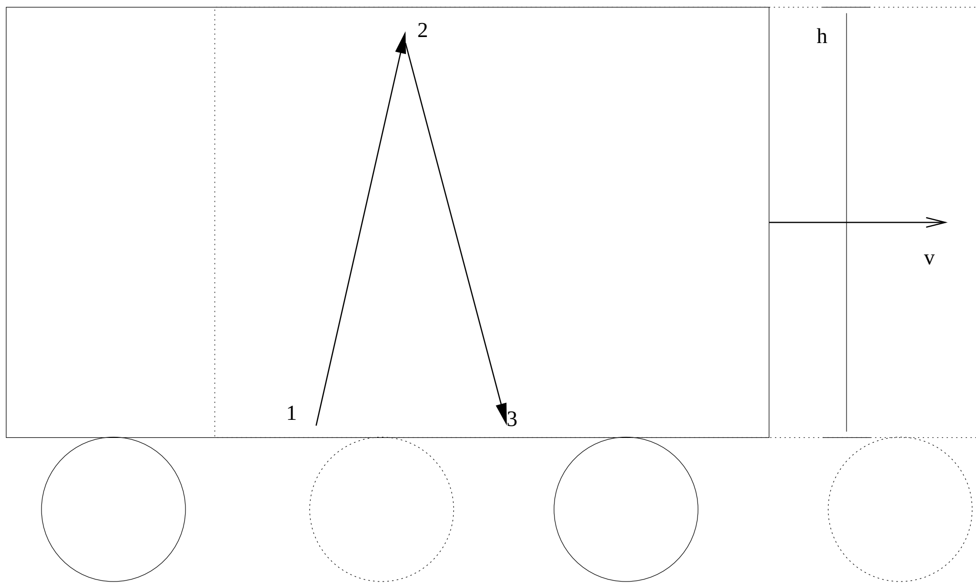
However, since the height of the train is h, the light must have traveled a distance of $2h$ between the beginning and end of the experiment, and thus:

$$\Delta t' = \frac{2h}{c}$$

All we've used here is the fact that light travels at the speed of light.

View from the unprimed frame:

Now, from the unprimed frame, the situation looks somewhat different. The detector has moved from when the laser was fired, and thus, from my perspective, the experiment looks like:



The solid train represents the position of the train at the beginning of the experiment, and the dotted line represents the position of the train at the end of the experiment. The detector must have moved a distance:

$$\Delta x = v\Delta t$$

from event "1" to "3". This is the same result we would have gotten from Galilean relativity. Well, except for one thing. We are no longer assuming that $\Delta t = \Delta t'$.

In this case, we note that the total trip length is:

$$d_{tot} = 2\sqrt{\left(\frac{v\Delta t}{2}\right)^2 + h^2}$$

In case it's not obvious where I got that, each light beam can be written as an x-component ($v\Delta t/2$, where Δt is the time observed by me), and a y-component, h. I'm just using the Pythagorean theorem to add them.

We also note that the light travel distance can be related to the full time interval between "1" and "3" via the relation:

$$\begin{aligned}
\Delta t &= \frac{d_{tot}}{c} \\
&= 2\sqrt{\left(\frac{v\Delta t}{2c}\right)^2 + \frac{h^2}{c^2}} \\
&= \sqrt{\left(\frac{v\Delta t}{c}\right)^2 + \left(\frac{2h}{c}\right)^2} \\
&= \sqrt{\left(\frac{v\Delta t}{c}\right)^2 + (\Delta t')^2}
\end{aligned}$$

where I've used $\Delta t' = 2h/c$ explicitly.

Interesting... I now have Δt in terms of $\Delta t'$. Squaring the expression, I get:

$$\begin{aligned}
\Delta t^2 &= \left(\frac{v\Delta t}{c}\right)^2 + (\Delta t')^2 \\
\left(1 - \frac{v^2}{c^2}\right)\Delta t^2 &= (\Delta t')^2 \\
\Delta t^2 &= \frac{(\Delta t')^2}{1 - v^2/c^2} \\
\Delta t &= \frac{\Delta t'}{\sqrt{1 - v^2/c^2}}
\end{aligned}$$

Thus, we've found an interesting relation:

$$\Delta t = \gamma \Delta t' + f(v)\Delta x' \tag{1}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \tag{2}$$

Note that I put a function $f(v)$ in there. Why? Because we don't know what happens if $\Delta x' \neq 0$. We'll figure it out eventually.

Also, by the same argument, since we found:

$$\Delta x = v\Delta t$$

then

$$\Delta x = v\gamma \Delta t' + g(v)\Delta x' \tag{3}$$

These equations need to be symmetric. In other words, there shouldn't be anything which can distinguish between the "moving" (primed) frame and the "stationary" (unprimed frame). To you, I should just appear to be moving to the left. And thus, we should have the relations:

$$\Delta x' = -v\gamma \Delta t + g(-v)\Delta x \tag{4}$$

and

$$\Delta t' = \gamma \Delta t + f(-v)\Delta x \tag{5}$$

To figure out what $f(v)$ and $g(v)$ are, we first note, that by definition, $\Delta x = \Delta x$. Thus, plugging equations 4 and 5 into 3 we get:

$$\begin{aligned}
\Delta x &= v\gamma(\gamma\Delta t + f(-v)\Delta x) + g(v)(-v\gamma\Delta t + g(-v)\Delta x) \\
&= (v\gamma f(-v) + g(v)g(-v))\,\Delta x + \left(v\gamma^2 - v\gamma g(v)\right)\Delta t
\end{aligned}$$

Since the left has to equal the right, the terms in front of Δx have to add to 1, and the terms in front of Δt have to add to 0. Thus:

$$\begin{aligned}
v\gamma^2 &= v\gamma g(v) \\
v\gamma f(-v) &= 1 - g(v)g(-v)
\end{aligned}$$

The first of these yield:

$$g(v) = g(-v) = \gamma$$

while plugging in gamma (after a bit of algebra) yields:

$$f(v) = \gamma v/c^2$$

We (finally!) have our Lorentz transforms:

$$\begin{aligned}
\Delta x' &= \gamma\Delta x - v\gamma\Delta t && (6)\\
\Delta t' &= \gamma\Delta t - \frac{v}{c^2}\gamma\Delta x && (7)\\
\Delta x &= \gamma\Delta x' + v\gamma\Delta t' && (8)\\
\Delta t &= \gamma\Delta t' + \frac{v}{c^2}\gamma\Delta x' && (9)
\end{aligned}$$

Note that if $\gamma \simeq 1$, then these are simply $\Delta x' = \Delta x - v\Delta t$, and $\Delta t' = \Delta t$, just as with the Galilean transforms.

Moreover, as with our Galilean transforms, we can combine these to figure out how velocity transforms:

$$\begin{aligned}
u' &= \frac{u-v}{1-\frac{uv}{c^2}} && (10)\\
u &= \frac{u'+v}{1+\frac{u'v}{c^2}} && (11)\\
& && (12)
\end{aligned}$$

<u>Some Examples</u>

A Lightbeam:

Consider what happens if you shine a flashlight ($u' = c$) on your train which is moving, say, at half the speed of light ($v = c/2$). The speed *I* observe the lightbeam to be traveling is:

$$\begin{aligned}
u &= \frac{c + c/2}{1 + \frac{c/2\ c}{c^2}}\\
&= \frac{1.5c}{1.5}\\
&= c
\end{aligned}$$

A Meter Stick:

Now, what happens if you have a meter stick in your ship (which is lying on the floor in the $\hat{i}$ direction)? You observe it as having a length, of course, of 1 m.

I, however, notice something different. Since I'm measuring the length at a particular moment, I measure $\Delta t = 0$ (the front and back are measured at the same time). Thus, according to equation 7, $\Delta x' = \gamma\Delta x$. Or, if I am measuring the length of the rod in my frame:

$$\Delta x = \frac{\Delta x'}{\gamma} = \frac{L}{\gamma}$$

Since γ is always greater than or equal to 1, I always measure moving meter sticks to be shorter than stationary ones. Not just meter sticks – everything. You, your ship, your control panels, and so on.

This is known as **length contraction**. Now here's the wacky thing – because of the symmetry of the equations, you will find that a meter stick in my frame (again, oriented along the x-axis) looks short to you.

A Clock:

What about a clock? Well, imagine that a clock ticks on your ship $\Delta t' = 1s$. For you, the two ticks of the clock happen in the same place $\Delta x' = 0$. Thus, according to equation 10, we have $\Delta t = \gamma \Delta t'$. In other words your clock (and your heartbeat, and the oscillations of your molecules and everything else, indeed time itself) seems to run slow on your ship.

But again – to you, it appears that my clock (and heart, etc.) are running slow!

This is known as **time dilation**.

A Spacetime Example

Here's a snapshot of a stationary lab in which a light beam is bouncing back and forth as seen from within the lab, and also as seen from an observer moving at 1/2 the speed of light: