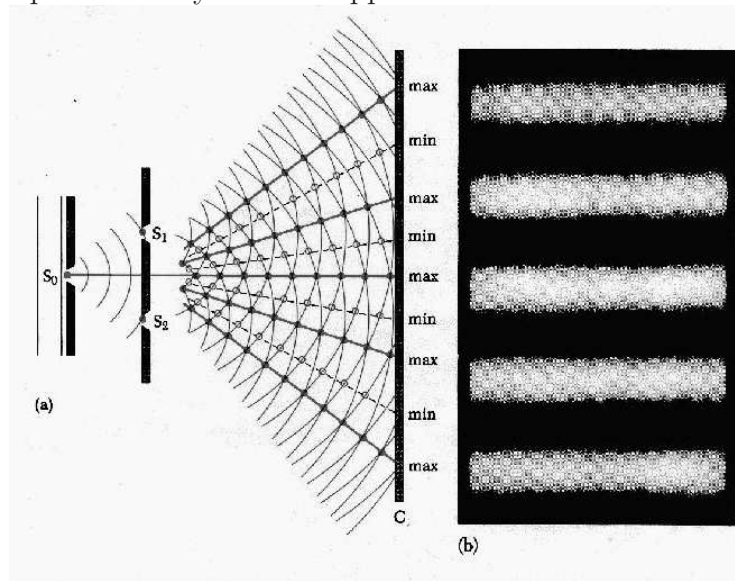


Waves and Interference

In much of our class, we have taken the view that everything in the world behaves more or less like a particle. Even photons, which we describe by their wavelength (note the “wave”) are said to have a particular position and velocity. We’ve spent very little time talking about the wavelike properties of either light, or, say, electrons.

What do we mean that light has a particular wavelength? Consider the classic “two slit” experiment, in which a beam of monochromatic (all 1 wavelength) light is shined at two slits in an otherwise opaque boundary. What happens?



The light, behaving like a wave, emanates out from each of the slits independently. What does this mean, from a mathematical point of view? Well, remember that light is the carrier of the electrical field. Thus, the electrical field (and similarly the magnetic field), might, at a distance, x_1 , from the first slit, has a value:

$$E_1(x_1, t) = E_0 \cos(kx_1 - \omega t) \quad (1)$$

where k (for now) is an unknown constant, and ω is the angular frequency of oscillations.

We might ask, “how bright would the light appear on the wall if there were only 1 slit?”

Well, the *intensity* of the light is simply the square of the electrical field:

$$I(x_1, t) = E_0^2 \cos^2(kx_1 - \omega t)$$

However, if we average this over time, we find that the \cos^2 averages to $1/2$, and thus the average intensity is just:

$$\langle I(x_1) \rangle = \frac{E_0^2}{2}$$

In other words, except for the fact that the light is more or less pointed forward, and thus the brightness will fade off near the edges, the brightness would form a constant pattern.

Now, what happens when we add a second slit? We get:

$$E_{tot} = E_0 [\cos(kx_1 - \omega t) + \cos(kx_2 - \omega t)]$$

$$\begin{aligned}
 I_{tot} &= E_0^2 \left[\cos^2(kx_1 - \omega t) + \cos^2(kx_2 - \omega t) + 2 \cos(kx_1 - \omega t) \cos(kx_2 - \omega t) \right] \\
 &= E_0^2 \left[\cos^2(kx_1 - \omega t) + \cos^2(kx_2 - \omega t) + \cos(k(x_1 - x_2)) + \cos(k(x_1 + x_2) - 2\omega t) \right] \\
 \langle I_{tot} \rangle &= E_0^2 [1 + \cos(k(x_1 - x_2))]
 \end{aligned}$$

I've used a bunch of trig identities here, but nothing you haven't seen before.

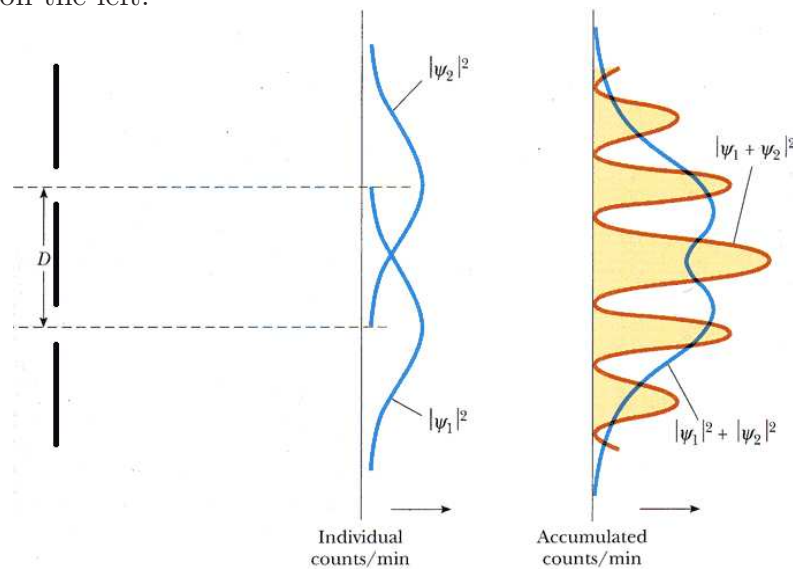
Now, what does this mean? It means that if $k(x_1 - x_2) = 2\pi n$ ($n = \dots -2, -1, 0, 1, 2, 3, \dots$) then the waves add "constructively" and you see a bright spot. If, on other hand, $k(x_1 - x_2) = 2\pi(n + 1/2)$, then the cos function gives -1, and the average intensity is zero. You see a dark spot.

Remember our basic method: We had several waves. We added them together, and we squared the result. Thus, the *square* of the wave-function tells us something important.

It works with electrons, too

Now, what happens if instead of firing a beam of light, we fire a beam of electrons through the two slits. Imagine further that we have a scintillator on the far wall, so that the brightness of the wall is proportional to the number of electrons striking it. What pattern would we expect?

Surely, you might think, electrons behave like particles, and thus, you might expect that electrons might simply pass through one slit or the other, and form a double peaked pattern like that seen on the left:



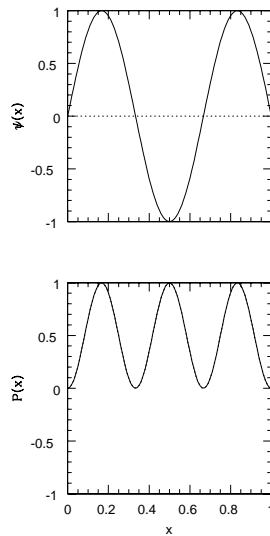
But, in fact, electrons behave exactly like light, and thus have the multiply peaked behavior as seen on the right.

What does this mean? It means that electrons (and all other particles as well) have a wave-like pattern not unlike that of light. We call this wave-function $\psi(x)$, and it has the property that:

$$P(x)dx = |\psi(x)|^2 dx \quad (2)$$

For example, consider a particle trapped in a 1-d box. It might have a particular wave-

function (never mind how we know this), which looks like the top panel shown here:



What does this mean? It means that if you were to look in such a box, you might find (with equal probability) the electron to be “near” one of three spots. There are certain places (where the probability is 0, for example), where you’d never find it.

One caveat: once you look at the electron or observe it in any way, you will totally change its wave-function. After all, you *know* where it is! What you’ll find is that, by the uncertainty principle, you now don’t know the momentum at all.

The Time-Independent Schroedinger Equation

So how do we figure out what wave-functions a particle is allowed to have? Well, we need to solve the Schroedinger Wave equation. In general this can be tricky, but there are some specific cases where it’s fairly easy. First, the equation itself is:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = (E - V(x))\psi \quad (3)$$

How you should read this is:

“If the particle exists in some known potential field ($V(x)$), and has a fixed energy, E (just a number), then this differential equation for the wave-function must be satisfied.”

Consider the easy case: a free particle. In other words, $V(x) = 0$. We have:

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{2mE}{\hbar^2} \psi$$

You have *totally* seen this equation before! It’s simply:

$$\psi(x) = \psi_0 \cos(kx)$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

or, putting it another way:

$$E = \frac{\hbar^2 k^2}{2m}$$

Hmm... this looks familiar. It seems like k (the “wavenumber”) is related to the momentum via the relationship:

$$p = \hbar k \quad (4)$$

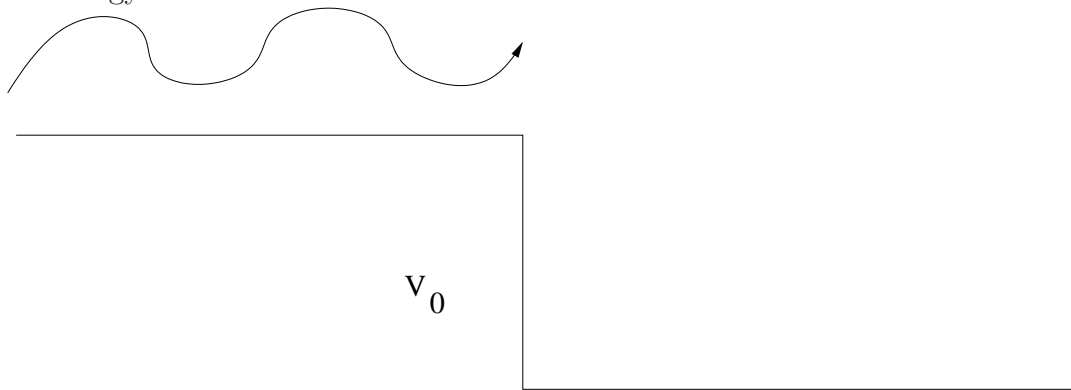
Note, by the way, that $k = 2\pi/\lambda$, and thus:

$$\lambda = \frac{h}{p} \quad (5)$$

In other words, all moving things have a wavelength, the “DeBroglie Wavelength.” It’s only when the DeBroglie wavelength is large compared to, say, the distance between the slits, that we care about the quantum effects.

A sudden “fall”

What happens if an electron is flying through space, and all of a sudden, it drops to a lower potential energy? Such a situation is shown below:



To the left of the boundary, the equation yields:

$$k_{left} = \sqrt{\frac{2mE}{\hbar^2}}$$

or

$$p_{left} = \sqrt{2mE}$$

while on the right, the equation yields:

$$p_{right} = \sqrt{2m(E + V_0)}$$

Or, to put it another way:

$$\frac{p_{right}^2}{2m} = \frac{p_{left}^2}{2m} + V_0$$

Which is basically just our conservation of energy equation!