Chapter 2

The Momentum Principle

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In this chapter we introduce the Momentum Principle, the first of three fundamental principles of mechanics which together make it possible to predict and explain a very broad range of real-world phenomena (the other two are the Energy Principle and the Angular Momentum Principle). In Chapter 1 you learned how to describe positions and motions in 3D, and we discussed the notion of "interaction" where change is an indicator that an interaction has occurred. We introduced the concept of momentum as a quantity whose change is related to the amount of interaction occurring. The Momentum Principle makes a quantitative connection between amount of interaction and change of momentum.

The major topics in this chapter are:

- The Momentum Principle, relating momentum change to interaction
- Force as a quantitative measure of interaction
- The concept of a "system" to which to apply the Momentum Principle

2.1 The Momentum Principle

Newton's first law of motion, "the stronger the interaction, the bigger the change in the momentum," states a qualitative relationship between momentum and interaction. The Momentum Principle restates this relation in a powerful quantitative form that can be used to predict the behavior of objects. The validity of the Momentum Principle has been verified through a very wide variety of observations and experiments. It is a summary of the way interactions affect motion in the real world.

THE MOMENTUM PRINCIPLE

$$\Delta\vec{p} = \vec{F}_{net}\Delta t$$

(for a short enough time interval Δt)

In words: change of momentum (the effect) is equal to the net force acting on an object times the duration of the interaction (the cause).

As usual, the capital Greek letter delta (Δ) means "change of" (something), or "final minus initial." The "net" force $\vec{F}_{net}$ is the vector sum of all the forces acting on an object. We will study forces in detail in this chapter. Examples of forces include

- the repulsive electric force a proton exerts on another proton
- the attractive gravitational force the Earth exerts on you
- the force that a compressed spring exerts on your hand
- the force on a spacecraft of expanding gases in a rocket engine
- the force of the air on the propeller of an airplane or swamp boat

Time interval short enough

We require a "short enough time interval" for the Momentum Principle to be valid, in the sense that the net force shouldn't change very much during the time interval. If the net force hardly changes during the motion, we can use a very large time interval. If the net force changes rapidly, we need to use a series of small time intervals for accuracy. This is the same issue we met with the position update relation, $\vec{r}_f = \vec{r}_i + \vec{v}_{avg}\Delta t$, where we need to use a short enough time interval that the velocity isn't changing very much, or else we need to know the average velocity during the time interval.

Since $\Delta\vec{p} = \vec{p}_f - \vec{p}_i = \vec{F}_{net}\Delta t$ ("final minus initial"), we can rearrange the Momentum Principle like this:

UPDATE FORM OF THE MOMENTUM PRINCIPLE

$$\vec{p}_f = \vec{p}_i + \vec{F}_{net}\Delta t \text{ (for a short enough time interval } \Delta t\text{)}$$

or, written out:

$$\langle p_{xf}, p_{yf}, p_{zf}\rangle = \langle p_{xi}, p_{yi}, p_{zi}\rangle + \langle F_x, F_y, F_z\rangle(\Delta t)$$

This update version of the Momentum Principle emphasizes the fact that if you know the initial momentum, and you know the net force acting during a "short enough" time interval, you can predict the final momentum. It's an interesting fact of nature that the x component of a force doesn't affect the y or z components of momentum, as you can see from these equations.

The Momentum Principle written in terms of vectors can be interpreted as three ordinary scalar equations, for components of the motion along the x, y, and z axes:

$$p_{xf} = p_{xi} + F_{\text{net, x}}\Delta t$$

$$p_{yf} = p_{yi} + F_{\text{net, y}}\Delta t$$

$$p_{zf} = p_{zi} + F_{\text{net, z}}\Delta t$$

Note how much information is expressed compactly in the vector form of this equation, $\vec{p}_f = \vec{p}_i + \vec{F}_{net}\Delta t$. In some simple situations, for example, if we know that the y and z components of an object's momentum are not changing, we may choose to work only with the x component of the momentum update equation.

The Momentum Principle has been experimentally verified in a very wide range of phenomena. We will see later that it can be restated in a very general form: the change in momentum of an object plus the change in momentum of its surroundings is zero (Conservation of Momentum). In this form the principle can be applied to all objects, from the very small (atoms and nuclei) to the very large (galaxies and black holes), though understanding these systems in detail requires quantum mechanics or general relativity.

Historically, the Momentum Principle is often called "Newton's second law of motion." We will refer to it as the Momentum Principle to emphasize the key role played by momentum in physical processes.

You are already familiar with change of momentum $\Delta\vec{p}$ and with time interval Δt. The new element is the concept of "force."

2.1.1 Force

Scientists and engineers employ the concept of "force" to quantify interactions between two objects. The net force, the vector sum of all the forces acting on an object, acting for some time Δt causes changes of momentum (Figure 2.1). Like momentum or velocity, force is described by a vector, since a force has a magnitude and is exerted in a particular direction. Measuring the magnitude of the velocity of an object (in other words, measuring its speed) is a familiar task, but how do we measure the magnitude of a force?

$$\Delta\vec{\mathbf{p}} = \vec{\mathbf{F}}_{net}\Delta t$$

Figure 2.1 The bigger the net force, the greater the change of momentum.

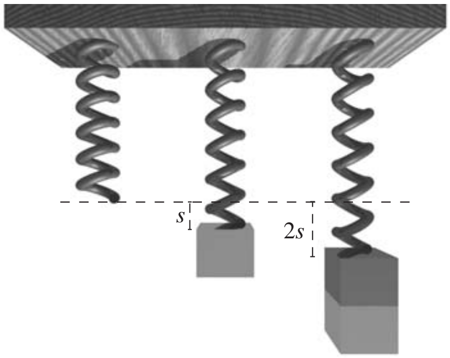


Figure 2.2 Stretching of a spring is a measure of force.

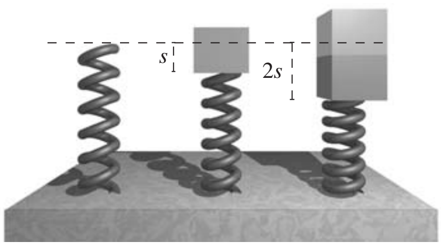


Figure 2.3 Compression of a spring is also a measure of force.

A simple way to measure force is to use the stretch or compression of a a spring. In Figure 2.2 we hang a block from a spring, and note that the spring is stretched a distance *s*. Then we hang two such blocks from the spring, and we see that the spring is stretched twice as much. By experimentation, we find that any spring made of the same material and produced to the same specifications behaves in the same way. Similarly, we can observe how much the spring compresses when the same blocks are supported by it. We find that one block compresses the spring by the same distance $|s|$, and two blocks compress it by $2|s|$ (Figure 2.3). (Compression is considered negative stretch, because the length of the spring decreases.)

We can use a spring to make a scale for measuring forces, calibrating it in terms of what force is required to produce a given stretch. The SI unit of force is the "newton," abbreviated as "N." One newton is a rather small force. A newton is approximately the downward gravitational force of the Earth on a small apple, or about a quarter of a pound. If you hold a small apple at rest in your hand, you apply an upward force of about one newton, compensating for the downward pull of the Earth.

The spring force law

A "force law" describes mathematically how a force depends on the situation. In this chapter we'll learn about various force laws, including the gravitational force law and the electric force law. For a spring, the magnitude of the force exerted by a spring on an object attached to the spring is given by the following force law:

THE SPRING FORCE LAW (MAGNITUDE)

$$|\vec{F}_{\text{spring}}| = k_s|s|$$

$|s|$ is the absolute value of the stretch; formally:

$$s = \Delta L = L - L_0$$

L_0 is the length of the relaxed spring
L is the length of the spring when stretched or compressed
k_s is the "spring stiffness"

The constant k_s is a positive number, and is a property of the particular spring: the stiffer the spring, the larger the spring stiffness, and the larger the force needed to stretch the spring. Note that *s* is positive if the spring is stretched ($L > L_0$) and negative if the spring is compressed ($L < L_0$).

We will see later that this equation can be rewritten as a vector equation, which gives both the magnitude and the direction of the force.

In the following, be sure to work through the "stop and think" activities before reading ahead. To learn the material, you need to engage actively with the questions posed here, not just read passively.

? Suppose a certain spring has been calibrated so that we know that its spring stiffness k_s is 500 N/m. You pull on the spring and observe that it is 0.01 m (1 cm) longer than it was when relaxed. What is the magnitude of the force exerted by the spring on your hand?

The force law gives $|\vec{F}_{\text{spring}}| = (500\ \text{N/m})|{+0.01\ \text{m}}| = 5\ \text{N}$. Note that the total length of the spring doesn't matter; it's just the amount of stretch or compression that matters.

? Suppose that instead of pulling on the spring, you push on it, so the spring becomes shorter than its relaxed length. If the relaxed length of the spring is 10 cm, and you compress the spring to a length of 9 cm, what is the magnitude of the force exerted by the spring on your hand?

The stretch of the spring in SI units is

$$s = L - L_0 = (0.09\ \text{m} - 0.10\ \text{m}) = -0.01\ \text{m}\,.$$

The force law gives

$$\left|\vec{\mathrm{F}}_{\text{spring}}\right| = (500\ \text{N/m})|-0.01\ \text{m}| = 5\ \text{N}\,.$$

The magnitude of the force is the same as in the previous case. Of course the direction of the force exerted by the spring on your hand is now different, but we would need to write a full vector equation to incorporate this information.

Reciprocity

In the preceding example, you pushed on a spring, compressing it, and we calculated the force exerted by the compressed spring on your hand. Of course, your hand has to exert a force on the spring in order to keep it compressed. It turns out that the force exerted by your hand on the spring is equal in magnitude (though opposite in direction) to the force exerted by the spring on your hand. This "reciprocity" of forces is a fundamental property of the electric interaction between the electrons and protons in your hand and the electrons and protons making up the spring. We will say more about the reciprocity of electric and gravitational forces later in this chapter.

Ex. 2.1 You push on a spring whose stiffness is 11 N/m, compressing it until it is 2.5 cm shorter than its relaxed length. What is the magnitude of the force the spring now exerts on your hand?

Ex. 2.2 A spring is 0.17 m long when it is relaxed. When a force of magnitude 250 N is applied, the spring becomes 0.24 m long. What is the stiffness of this spring?

Ex. 2.3 The spring in the previous exercise is now compressed so that its length is 0.15 m. What magnitude of force is required to do this?

2.1.2 Impulse

The amount of interaction affecting an object includes both a measure of the strength of the interaction expressed as the net force $\vec{\mathrm{F}}_{\text{net}}$ and of the duration Δt of the interaction. Either a bigger force, or a longer application of the force, will cause more change of momentum.

The product of a force and a time interval is called "impulse" (Figure 2.4):

DEFINITION OF IMPULSE

$$\text{Impulse} \equiv \vec{\mathrm{F}}\Delta t \ \ \text{(for small enough } \Delta t\text{)}$$

Impulse has units of N·s (newton-seconds)

With this definition of impulse we can state the Momentum Principle in words like this:

The change of momentum of an object is equal to the net impulse applied to it.

$$\Delta\vec{\mathbf{p}} = \vec{\mathbf{F}}_{\text{net}}\Delta t$$

Figure 2.4 Net impulse: net force times duration of the interaction. Net impulse changes momentum.

? A constant net force $\langle 3, -5, 4\rangle$ N acts on an object for 10 s. What is the net impulse applied to the object?

$$\text{impulse} = \vec{F}_{\text{net}}\Delta t = \langle 3, -5, 4\rangle \text{ N} \cdot (10 \text{ s}) = \langle 30, -50, 40\rangle \text{ N} \cdot \text{s}$$

Ex. 2.4 A constant net force of $\langle -0.5, -0.2, 0.8\rangle$ N acts on an object for 2 minutes. What is the impulse applied to the object, in SI units?

2.1.3 Predictions using the Momentum Principle

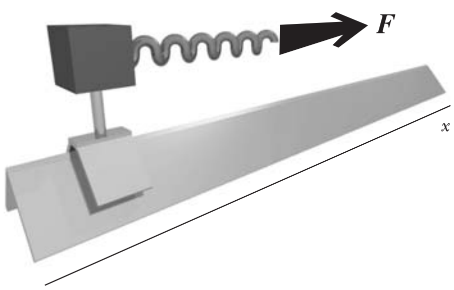


Figure 2.5 Apply a constant force to a block on a low-friction air track.

Experiments can easily be done to verify the Momentum Principle. Many introductory physics laboratories have air tracks like the one illustrated in Figure 2.5. The long triangular base has many small holes in it, and air under pressure is blown out through these holes. The air forms a cushion under the glider, allowing it to coast smoothly with very little friction. Suppose we place a block on a glider sitting on a long air track (Figure 2.5), and we attach a spring to it whose stiffness is 500 N/m, like the one discussed earlier. We will choose the x axis to point in the direction of the motion.

Suppose the block starts from rest, and you pull for 1 second with the spring stretched 4 cm, so that you know that you are pulling with a force of magnitude $(500 \text{ N/m})(0.04 \text{ m}) = 20 \text{ N}$ (you will have to move forward in order to keep the spring stretched).

? The block starts from rest, so $\vec{p}_i = \langle 0, 0, 0\rangle$ kg·m/s. What would the Momentum Principle predict the new momentum of the block to be after 1 second?

Since the friction force on the glider is negligibly small, the net force on the glider is just the force with which you pull. The Momentum Principle (update form) applied to the glider is:

$$\vec{p}_f = \langle 0, 0, 0\rangle \text{ kg·m/s} + \langle 20, 0, 0\rangle \text{ N}(1 \text{ s})$$

Since the net force is in the x direction, we know that the y and z components of the glider's momentum will not change, and we can work with just the x component of the momentum.

$$p_{xf} = p_{xi} + F_{\text{net, x}}\Delta t = 0 + (20 \text{ N})(1 \text{ s}) = 20 \text{ kg·m/s}$$

If you do the experiment, this is what you will observe.

You keep pulling for another second, but now with the spring stretched half as much, 2 cm, so you know that you're pulling with half the original force:

$$(500 \text{ N/m})(0.02 \text{ m}) = 10 \text{ N}$$

? What would the Momentum Principle predict the new x component of the momentum of the block to be now?

The Momentum Principle would predict the following, where we take the final momentum from the first pull and consider that to be the initial momentum for the second pull:

$$p_{xf} = p_{xi} + F_{\text{net, x}}\Delta t = (20 \text{ kg·m/s}) + (10 \text{ N})(1 \text{ s}) = 30 \text{ kg·m/s}$$

This is what is observed experimentally. Note that the effects of the interactions in the two 1-second intervals add; we add the two momentum changes.

Instead of varying the net force, we could try varying the duration of the interaction (Figure 2.6). Start over with the block initially at rest and pull for 2 seconds with the spring stretched 4 cm, so the force is 20 N.

$$\Delta\vec{p} = \vec{F}_{\text{net}}\Delta t$$

Figure 2.6 Duration of interaction: the longer the net force acts, the greater the change of momentum.

? The block starts from rest ($p_{xi} = 0$). What would the Momentum Principle predict the new *x* component of the momentum of the block to be?

$$p_{xf} = p_{xi} + F_{\text{net, x}}\Delta t = 0 + (20\text{ N})(2\text{ s}) = 40\text{ kg}\cdot\text{m/s}$$

Here the final *x* component of momentum was 40 kg·m/s after applying a force of 20 N for 2 s, whereas in the previous experiments we got 30 kg·m/s after applying a force of 20 N for 1 s plus a force of 10 N for 1s. In our calculations we can use big values of Δt as long as the force isn't changing much. When the force changed from 20 N for a second to 10 N for a second we had to treat the two time intervals separately.

Many different experiments have shown the validity of the Momentum Principle. If we use two springs to move the block, we find that it is indeed the vector sum of the two spring forces, the "net" force, that accounts for the change in momentum.

2.2 The superposition principle

? Drag this book across the table at constant velocity (constant speed and constant direction). You're applying a force, yet the momentum isn't changing. Explain briefly.

The force you apply to the book isn't the only force acting on the book. The table also applies a force, in the opposite direction, called a frictional force. This frictional force is due to collisions between atoms in the bottom layer of the book and atoms in the top layer of the table (see Figure 2.7 for an atomic picture of sliding friction). If the momentum isn't changing, it must be that the *net* force on the book is zero. This is why the Momentum Principle involves $\vec{F}_{\text{net}}$ rather than just the force you apply.

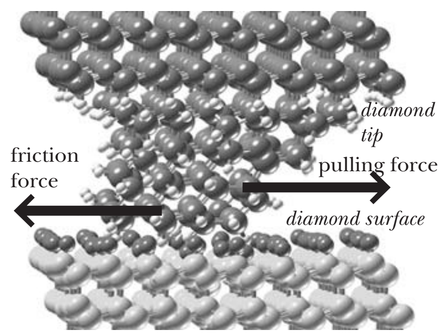


Figure 2.7 A computer simulation of sliding friction: a diamond tip is dragged along a diamond surface. Image courtesy of Judith A. Harrison, U.S. Naval Academy.

? What do you have to do to make the book go faster?

You have to apply a force that is bigger than the frictional force, so that the vector sum is nonzero.

? The Momentum Principle predicts that if the net force is zero, the momentum doesn't change but stays the same. Yet if you push the book across the table at high speed, then let go, the book doesn't keep moving but comes to a stop. Explain briefly.

After you let go, the net force on the book is just the frictional force of the table, which acts opposite to the momentum and makes the momentum decrease.

? What if you were in outer space, far from tables and other objects, and you pushed the book so that it was going fast, then let go. What would the book do?

In the absence of any forces (zero net force), the book would continue forever with the (vector) momentum you initially gave it. It would move in a straight line at constant speed: $\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t = \vec{p}_i$ if $\vec{F}_{\text{net}} = \vec{0}$.

Why it took so long to discover the Momentum Principle

In our everyday world friction between objects is common. As a result, for many centuries people quite naturally believed that a force was necessary to sustain motion, just as you saw when dragging the book across the table at constant speed. This made it hard to understand what made the planets keep moving. Galileo and Newton finally realized that it is the *net* force that matters, and that objects free of interactions just naturally keep moving, with no forces needed to keep them moving (Newton's first law of motion). This represented a major revolution in how humans viewed the world.

2.2.1 Net force

How do we calculate the net force acting on an object? It turns out to be surprisingly straightforward, as summarized by the superposition principle.

THE SUPERPOSITION PRINCIPLE

The net force on an object is the vector sum of the individual forces exerted on it by all other objects.

Each individual interaction is unaffected by the presence of other interacting objects.

The superposition principle is completely general. It has been found experimentally to apply to all kinds of interactions: gravitational, electromagnetic, and nuclear interactions. The implications of this principle are not always intuitively obvious. For example, this principle implies that the force exerted by the Sun on the Earth at a particular distance will always be the same, regardless of how many other planets there are that also interact with the Sun and the Earth. The presence of other objects and interactions does not block or change the interactions between each pair of objects. An interaction doesn't get "used up," and the interaction between one object and another is unaffected by the presence of a third object.

To take a silly example, hold a glass over a table and then let go (Figure 2.8). The glass falls, so evidently the table doesn't block the gravitational interaction with the Earth. In fact, the table adds a tiny additional gravitational force on the glass to that of the Earth, without changing the Earth's attraction for the glass.

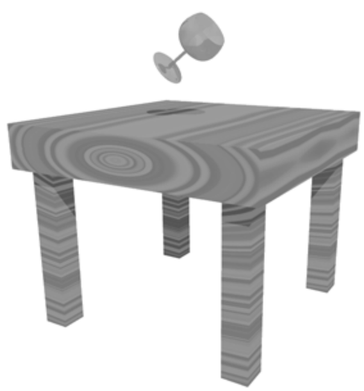


Figure 2.8 The presence of the table does not change the interaction of the glass with the Earth.

Ex. 2.5 A balloon experiences a gravitational force of $\langle 0, -0.05, 0\rangle$ N and a force due to the wind of $\langle -0.03, 0, 0.02\rangle$ N. What is the net force acting on the balloon?

Ex. 2.6 A sailboat sails straight toward an island at constant speed. What are all the forces acting on the boat? Is the net force zero or nonzero?

Ex. 2.7 You watch someone carry a heavy block left-to-right across the room, walking at constant speed. According to the Momentum Principle, should we conclude that the net force acting on the block is upward, toward the right, or zero?

2.3 System and surroundings

In order to use the Momentum Principle to predict motion, we must choose a "system" whose momentum change we will calculate. By the word "system" we mean some pieces of the Universe of interest to us. The rest of the Universe we call the "surroundings" (Figure 2.9).

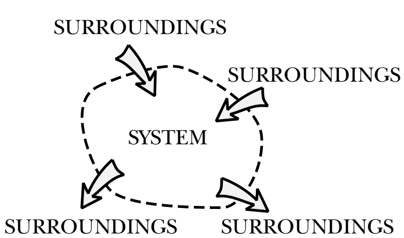


Figure 2.9 System and surroundings. Interactions in the form of impulses flow across the system boundary and change the system's momentum.

To predict the motion of a baseball through the air, it makes sense to choose the baseball as the system whose momentum changes. The surroundings include the Earth which exerts a gravitational force on our chosen system (the baseball), and the air, which exerts a force of air resistance against the moving ball (Figure 2.10). It is true that there are changes in the momentum of the Earth and the momentum of the air molecules due to interactions with the baseball, but once we decide to choose the baseball as the system of interest we don't have to pay attention to what happens to the momentum of objects in the surroundings. The role of the surroundings is described entirely by the forces they exert on our chosen system.

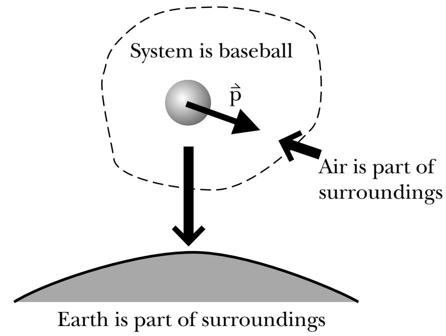


Figure 2.10 Choose a baseball as the system. The surroundings are the Earth and the air, which interact across the system boundary to change the momentum of the system.

Only external forces matter

It is an important rule that we do not include in the net force acting on the baseball any forces that the baseball exerts on itself. The atoms of the baseball do exert forces on each other, but as we'll discuss in more detail later, interatomic forces and gravitational force come in equal and opposite pairs (Figure 2.11). Therefore these "internal" force pairs, forces between pairs of atoms internal to our chosen system, add up to zero and can safely be ignored in calculating the net force that appears in the Momentum Principle, $\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t$. Only "external" forces matter, forces associated with interactions between our chosen system and objects in the surroundings.

Of course objects in the surroundings experience forces exerted by objects inside the system, but when we apply the Momentum Principle just to the system, we only care about the change of momentum of the system. Also, there are equal and opposite changes of momentum in the surroundings, but they typically don't interest us.

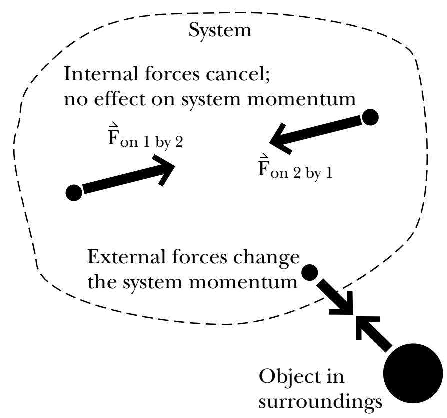


Figure 2.11 Internal force pairs cancel, so only external forces can change the momentum of a system.

Neglecting small effects

The surroundings of the system of the baseball includes the Sun, the Moon, Mars, etc. Do we have to consider all of the forces these objects exert on the baseball? In practice, no, because these forces are extremely small compared to the forces exerted on the system by the Earth and the air during the brief flight of the baseball. But if we were trying to plot an accurate course for a spacecraft going to Mars we would need to include even small forces exerted by other planets, because impulse is forces times time duration, $\vec{F}\Delta t$, and in the long time required to go to Mars even small forces can produce significant impulses, and significant changes in the momentum of the spacecraft.

Systems consisting of several objects

A system can consist of more than one object. For example, we might choose to consider a system consisting of the entire Solar System. The surroundings of this large system would include the rest of the Universe, including neighboring stars. The total momentum of the Solar System can change due to the gravitational forces exerted by stars, especially nearby stars.

Ex. 2.8 In Section 2.1.3 we applied the Momentum Principle to predict the motion of a glider on an air track. What did we choose as the system in this analysis? What objects were included in the surroundings?

2.4 Applying the Momentum Principle to a system

To apply the Momentum Principle to analyze the motion of a real-world system, several steps are required:

1. Choose a system, consisting of a portion of the Universe.
The rest of the Universe is called the surroundings.

2. List objects in the surroundings that exert significant forces on the chosen system, and make a labeled diagram showing the external forces exerted by the objects in the surroundings.

A labeled diagram (step 3) gives a physics view of the situation, and it defines symbols to use in writing an algebraic statement of the Momentum Principle.

3. Apply the Momentum Principle to the chosen system:

$$\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t$$

For each term in the Momentum Principle, substitute any values you know.

4. Apply the position update formula, if necessary:

$$\vec{r}_f = \vec{r}_i + \vec{v}_{\text{arg}}\Delta t$$

5. Solve for any remaining unknown quantities of interest.

6. Check for reasonableness (units, etc.).

2.4.1 Example: Position and momentum of a ball

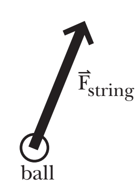
Inside a spaceship in outer space there is a small steel ball of mass 0.25 kg. At a particular instant, the ball is located at position $\langle 9, 5, 0\rangle$ m and has momentum $\langle -8, 3, 0\rangle$ kg·m/s. At this instant the ball is being pulled by a string, which exerts a net force $\langle 20, 50, 0\rangle$ N on the ball. What is the ball's approximate momentum and position 3 milliseconds later $(3\times 10^{-3}\ \text{s})$? What approximations or simplifying assumptions did you make in your analysis?

1. Choose a system

System: the steel ball

2. List external objects that interact with the system, with diagram

the string
(a circle represents the system)



3. Apply the Momentum Principle

$$\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t$$

$$\vec{p}_f = (\langle -8, 3, 0\rangle\ \text{kg·m/s}) + (\langle 20, 50, 0\rangle\ \text{N})(3\times 10^{-3}\ \text{s})$$

$$\vec{p}_f = (\langle -8, 3, 0\rangle\ \text{kg·m/s}) + (\langle 0.06, 0.15, 0\rangle\ \text{N·s})$$

$$\vec{p}_f = \langle -7.94, 3.15, 0\rangle\ \text{kg·m/s}$$

Assume that the force didn't change much during 3 ms. Or to put it another way, we assumed that a time interval of 3 ms was sufficiently short that we could update the momentum fairly well assuming a constant force during that short time interval.

4. Apply the position update formula

$$\vec{r}_f = \vec{r}_i + \vec{v}_{avg}\Delta t\text{, where } \vec{v} \approx \vec{p}/m \text{ since } v << c$$

$$\vec{r}_f = (\langle 9, 5, 0\rangle \text{ m}) + \frac{(\langle -7.94, 3.15, 0\rangle \text{ kg}\cdot\text{m/s})}{(0.25 \text{ kg})}(3\times10^{-3} \text{ s})$$

$$\vec{r}_f = (\langle 9, 5, 0\rangle \text{ m}) + (\langle -0.0953, 0.0378, 0\rangle \text{ m})$$

$$\vec{r}_f = \langle 8.905, 5.038, 0\rangle \text{ m}$$

5. There are no remaining unknowns

6. Check

Units check (momentum: kg·m/s and position: m)

We used the momentum at the end of the interval to update the position.

Note that at very high speeds, $\vec{v} \approx \vec{p}/m$ isn't valid for updating position. See page 98.

2.4.2 Example: A fan cart (1D, constant net force)

An easy way to arrange to apply a nearly constant force is to mount an electric fan on a cart (Figure 2.12). If the fan blows backwards, the interaction with the air pushes the cart forward with a nearly constant force, making the cart's momentum continually increase. Swamp boats used in the very shallow Florida Everglades are built in a similar way, with large fans on top of the boats propelling them through the swamp.

When a fan cart or boat gets going very fast, air resistance becomes important and at high speeds is as big as the propelling force, so that the net force becomes zero, at which point the momentum doesn't increase any more, and the cart or boat travels at constant speed. For simplicity we'll consider the motion at low speed, with negligible air resistance, so we can make the approximation that the net force is due solely to the fan and is nearly constant.

(Note that the y component of the net force is zero, because the downward gravitational force on the cart is exactly balanced by the upward force exerted by the track on the cart. In a later chapter we will examine the interaction between solid objects like the cart and the track in more detail.)

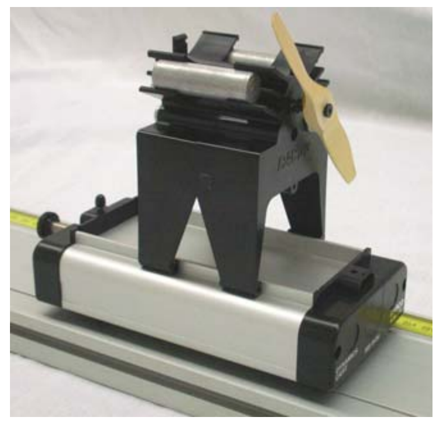


Figure 2.12 A fan cart on a track.

Predict new position and new momentum

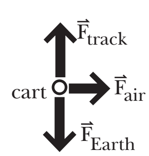
Suppose you have a fan cart whose mass is 400 grams (0.4 kg), and with the fan turned on, the net force acting on the cart, due to the air and friction with the track, is $\langle 0.2, 0, 0\rangle$ N and constant. You give the cart a shove, and you release the cart at position $\langle 0.5, 0, 0\rangle$ m with initial velocity $\langle 1.2, 0, 0\rangle$ m/s. What is the position of the cart 3 seconds later, and what is its momentum at that time?

1. Choose a system

System: the cart (including the fan)

2. List external objects that interact with the system, with diagram

the Earth, the track, the air
(represent the system by a circle)



Since $v \ll c$, we can use the approximation that $\vec{p} \approx m\vec{v}$.

Since the y component of the cart's momentum does not change, we know that the y component of the net force must be zero, and $|\vec{F}_{\text{track}}| = |\vec{F}_{\text{Earth}}|$.

We can use a large time interval Δt because the force isn't changing very much in either magnitude or direction.

3. Apply the Momentum Principle

$$\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t = \vec{p}_i + (\vec{F}_{\text{track}} + \vec{F}_{\text{Earth}} + \vec{F}_{\text{air}})(\Delta t)$$

$$\vec{p}_f \approx \langle (0.4\text{kg})(1.2\text{m/s}), 0, 0\rangle + \langle 0.2, (|\vec{F}_{\text{track}}| - |\vec{F}_{\text{Earth}}|), 0\rangle \text{N}(3\text{ s})$$

$$\vec{p}_f \approx \langle 1.08, 0, 0\rangle \text{ kg}\cdot\text{m/s}$$

Approximating the average velocity

In order to use the position update formula, we need the average velocity of the cart. How do we find $\vec{v}_{\text{avg}}$? We know initial and final values for $\vec{v}$:

$$\vec{v}_i = \langle 1.2, 0, 0\rangle \text{m/s}$$

$$\vec{v}_f \approx \frac{\vec{p}_f}{m} = \frac{\langle 1.08, 0, 0\rangle \text{ kg}\cdot\text{m/s}}{0.4\text{ kg}} = \langle 2.7, 0, 0\rangle \text{m/s}$$

Can we use these values to find $\vec{v}_{\text{avg}}$? You may have guessed that the average velocity is the "arithmetic" (pronounced "arithMETic") average:

$$\vec{v}_{\text{avg}} \approx \frac{(\vec{v}_i + \vec{v}_f)}{2}$$

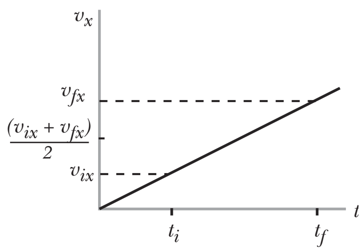


Figure 2.13 v_x is changing linearly with time, so the arithmetic average is equal to

The arithmetic average.is often a good approximation, but it is not always exactly equal to the average velocity $\vec{v}_{\text{avg}} = \Delta\vec{r}/\Delta t$. The arithmetic average does lie between the two extremes. For example, the arithmetic average of 6 and 8 is (6+8)/2 = 14/2 = 7, halfway between 6 and 8. See Figure 2.13.

The arithmetic average does not give the true average velocity unless the velocity is changing at a constant rate, which is the case only if the net force is constant, as it happens to be for a fan cart. For example, if you drive 50 mi/hr for four hours, and then 20 mi/hr for an hour, you go 220 miles, and your average speed is (220 mi)/(5 hr) = 44 mi/hr, whereas the arithmetic average is (50+20)/2 = 35 mi/hr. In situations where the force is not constant, we have to choose short enough time intervals that the velocity is nearly constant during the brief Δt (see Figure 2.14).

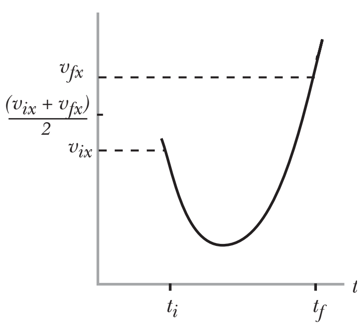


Figure 2.14 v_x is not changing linearly with time. In this case the arithmetic average is much higher than the average value of v_x.

APPROXIMATE AVERAGE VELOCITY

$$\vec{v}_{\text{avg}} \approx \frac{(\vec{v}_i + \vec{v}_f)}{2} \quad (\text{if } v << c)$$

exactly true *only* if $\vec{v}$ changes at a constant rate ($\vec{F}_{\text{net}}$ constant)

The proof (which is more complicated than one might expect) is given in optional Section 2.11 at the end of this chapter.

The net force on the cart is constant, so this calculation of average velocity gives the correct value.

4. Apply the position update formula

$$\vec{v}_{\text{avg}} = \frac{(\vec{v}_i + \vec{v}_f)}{2} = \left\langle \frac{(1.2+2.7)}{2}, \frac{(0+0)}{2}, \frac{(0+0)}{2}\right\rangle \frac{\text{m}}{\text{s}}$$

$$\vec{v}_{\text{avg}} = \langle 1.95, 0, 0\rangle \text{m/s}$$

$$\vec{r}_f = \vec{r}_i + \vec{v}_{\text{arg}}\Delta t = \langle 0.5, 0, 0\rangle \text{m} + \langle 1.95, 0, 0\rangle \frac{\text{m}}{\text{s}}(3\text{s})$$

$$\vec{r}_f = \langle 6.35, 0, 0\rangle \text{m}$$

5. There are no remaining unknowns

6. Check

Position: m (correct units); $x_f > x_i$, as it should be.

Ex. 2.9 A hockey puck is sliding along the ice with nearly constant momentum $\langle 10, 0, 5\rangle$ kg · m/s when it is suddenly struck by a hockey stick with a force $\langle 0, 0, 2000\rangle$ N that lasts for only 3 milliseconds (3×10^{-3} s). What is the new momentum of the puck?

Ex. 2.10 You were driving a car with velocity $\langle 25, 0, 15\rangle$ m/s. You quickly turned and braked, and your velocity became $\langle 10, 0, 18\rangle$ m/s. The mass of the car was 1000 kg. What was the (vector) change in momentum $\Delta\vec{p}$ during this maneuver? Pay attention to signs. What was the (vector) impulse applied to the car by the ground?

Ex. 2.11 In the previous exercise, if the maneuver took 3 seconds, what was the average net (vector) force $\vec{F}_{net}$ that the ground exerted on the car?

Ex. 2.12 A truck driver slams on the brakes and the momentum changes from $\langle 9\times10^4, 0, 0\rangle$ kg · m/s to $\langle 5\times10^4, 0, 0\rangle$ kg · m/s in 4 seconds due to a constant force of the road on the wheels of car. As a vector, write the force exerted by the road.

Ex. 2.13 At a certain instant, a particle is moving in the $+x$ direction with momentum +10 kg·m/s. During the next 0.1 s, a constant force $\langle -6, 3, 0\rangle$ N acts on the particle. What is the momentum of the particle at the end of this 0.1 s interval?

2.4.3 Example: A thrown ball (2D, constant net force)

A second example is the prediction of the motion of a ball thrown through the air. The Earth pulls down on the ball with a gravitational force, and the air pushes against the ball as the ball runs into air molecules. When a high-density object is thrown at low speed, this "air resistance" force is rather small compared to the gravitational force, so we may be able to neglect air resistance.

A low-density object such as a styrofoam ball experiences air resistance that is comparable to the small gravitational force on the ball, so air resistance is important unless the styrofoam ball is moving very slowly (air resistance is small at low speeds and big at high speeds, as you may have experienced if you put your hand out the window of a car). At low speeds a baseball, which has a fairly high density, moves with negligible air resistance. But at the speed that a professional pitcher can throw a baseball (about 90 mi/hr or 40 m/s), a baseball goes only about half as far in air as it would in a vacuum, because air resistance is large at this high speed (Figure 2.15).

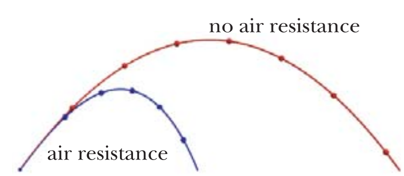


Figure 2.15 The trajectory of a baseball thrown at high speed, ignoring air resistance (top curve) and including the effect of air resistance (bottom curve). The dots indicate the ball's position at equal time intervals.

We'll analyze the flight of a ball with the assumption that we can neglect air resistance, which means we are dealing with a high-density object at low speed, or motion in a vacuum. Later we'll develop techniques that make it possible to predict the motion of a ball when air resistance is significant.

Near the Earth's surface, every kilogram of matter is attracted toward the center of the Earth by a force of approximately 9.8 N. The Earth's gravitational pull on a 2 kg block is (2 kg)(9.8 N/kg), or 19.6 N. In general, the Earth exerts a force on a mass m like this:

$$\text{Near the Earth's surface, } \left|\vec{F}_{grav}\right| \approx mg\text{, where } g = +9.8 \text{ N/kg}$$

We'll have more to say about gravitational forces later in this chapter, but this is sufficient for analyzing the motion of a thrown ball.

Predict new velocity and new position of a thrown ball

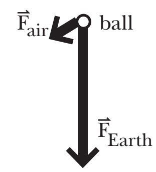
You throw the ball so that just after it leaves your hand at location $\langle x_i, y_i, 0\rangle$ it has velocity $\langle v_{xi}, v_{yi}, 0\rangle$, with no component in the z direction. Now that it has left your hand, and we're neglecting air resistance, the net force at all times is $\langle 0, -mg, 0\rangle$, since the Earth's gravitational force acts downward, toward the center of the Earth, and we normally choose our axes so y points up. Predict the velocity and position of the ball after a time Δt. .

1. Choose a system

System: the ball

2. List external objects that interact with the system, with diagram

the Earth, the air
(represent the system by a circle)



We make the approximation that air resistance is negligible compared to the gravitational force.

3. Apply the Momentum Principle

$v \ll c$ so $\vec{p} \approx m\vec{v}$

$$\vec{p}_f = \vec{p}_i + \vec{F}_{net}\Delta t$$

$$m\langle v_{xf}, v_{yf}, 0\rangle = m\langle v_{xi}, v_{yi}, 0\rangle + \langle 0, -mg, 0\rangle\Delta t$$

Divide both sides of the equation by m

$$\begin{aligned}\langle v_{xf}, v_{yf}, 0\rangle &= \langle v_{xi}, v_{yi}, 0\rangle + \langle 0, -g, 0\rangle\Delta t\\ &= \langle (v_{xi} + 0\Delta t), (v_{yi} + -g\Delta t), (0 + 0(\Delta t))\rangle\\ &= \langle v_{xi}, (v_{yi} + -g\Delta t), 0\rangle\end{aligned}$$

The x and z components of velocity are not changing; this makes sense because the net force has only a y component. The y component of velocity is decreasing continuously; this makes sense because the gravitational force on the ball by the Earth affects the y component of the ball's velocity.

4. Apply the position update formula

$$\vec{v}_{arg} = \langle\left(\frac{v_{xi} + v_{xi}}{2}\right), \frac{(v_{yi} + (v_{yi} + -g\Delta t))}{2}, \frac{(0 + 0)}{2}\rangle$$

$$\vec{v}_{arg} = \langle v_{xi}, (v_{yi} - \tfrac{1}{2}g\Delta t), 0\rangle$$

$$\vec{r}_f = \vec{r}_i + \vec{v}_{arg}\Delta t$$

$$\langle x_f, y_f, 0\rangle = \langle x_i, y_i, 0\rangle + \langle v_{xi}, (v_{yi} - \tfrac{1}{2}g\Delta t), 0\rangle\Delta t$$

$$\langle x_f, y_f, 0\rangle = \langle (x_i + v_{xi}\Delta t), (y_i + v_{yi}\Delta t - \tfrac{1}{2}g(\Delta t)^2), 0\rangle$$

Alternatively:

$$\begin{aligned}x_f &= x_i + v_{xi}\Delta t\\ y_f &= y_i + v_{yi}\Delta t - \tfrac{1}{2}g(\Delta t)^2\\ z_f &= 0\end{aligned}$$

5. There are no remaining unknowns

6. Check

The units check for the final position. Note that g is N/kg, so that $g\Delta t$ is N·s/kg, units of impulse/kg, which is kg·m/s/kg, or m/s.

? Under what circumstances can we use this result to predict the trajectory of an object?

We assumed that air resistance was negligible, that the object was traveling at a speed much less than the speed of light, and that the net force was constant. If any of these three conditions are not met, then these results do not apply, and using them will give a wrong answer.

Using this result

If a ball of mass 90 g is initially on the ground, at location $\langle 0, 0, 0\rangle$ m , and you kick it with initial velocity $\langle 3, 7, 0\rangle$ m/s , where will the ball be half a second later?

$$x_f = 0 + (3\text{m/s})(0.5\text{ s}) = 1.5\text{ m}$$
$$y_f = 0 + (7\text{m/s})(0.5\text{ s}) - (9.8\text{N/kg})(0.5\text{ s})^2 = 1.05\text{ m}$$

? Can we use these equations to find the location of the ball 10 seconds after you kick it?

No. Our result would be that the ball was far underground:

$$y_f = (7\text{m/s})(10\text{ s}) - (9.8\text{N/kg})(10\text{ s})^2 = -910\text{ m}$$

which is not physically reasonable! (The ball would have hit the ground and stopped before 10 seconds had passed.)

? What should you remember from the preceding example?

If you understand the basic method, you can reproduce the specific results quickly and accurately. Just memorizing the results won't help you much, because you won't really understand what they mean or when they can be used.

For concreteness we analyzed the flight of a thrown ball, but this same analysis would work for any situation where an object moves in two dimensions under the influence of a constant net force. For example, just by changing the constant y component of the net force to be something other than $-mg$, we could analyze the two-dimensional motion of a swamp boat (if the direction of the thrust of the propeller doesn't change, and we can neglect friction with the water), or to the two-dimensional motion of an electron between two large charged plates. However, in these cases, the resulting equations would be slightly different, because the mass of the object would appear in the result (the m cancels only in the case of a force that is proportional to m, such as the gravitational force).

Graphs of the motion.

In Figure 2.16 we shows graphs of position and velocity components vs. time, and the actual path (y vs. x) of the ball. The first graph, v_x vs. t, is simply a horizontal line, because v_x doesn't change, since there is no x component of force. The graph of x is a straight line (second graph), rising if v_x is positive. Note that the slope of the x vs. t graph is equal to v_x.

The graph of v_y is a falling straight line (third graph), because the y component of the force is $-mg$, which constantly makes the y component of momentum decrease. At some point the y component of momentum decreases to zero, at the top of the motion, after which the ball heads downward, with negative v_y. The graph of y vs. time t is an inverted parabola (fourth graph), since the equation for y is a quadratic function in the time.

Note that the slope of the y vs. t graph (the fourth graph) at any time is equal to v_y at that time. In particular, when the slope is zero (at the maximum y), v_y is momentarily equal to zero. Before that point the slope is positive, corresponding to $v_y > 0$, and after that point the slope is negative, corresponding to $v_y < 0$.

The actual path of the ball, the graph of y vs. x, is also an inverted parabola (the bottom graph). Since x increases linearly with t, whether we plot y vs. t or y vs. x we'll see a similar curve. The scale factor along the horizontal axis is different, of course (meters instead of seconds).

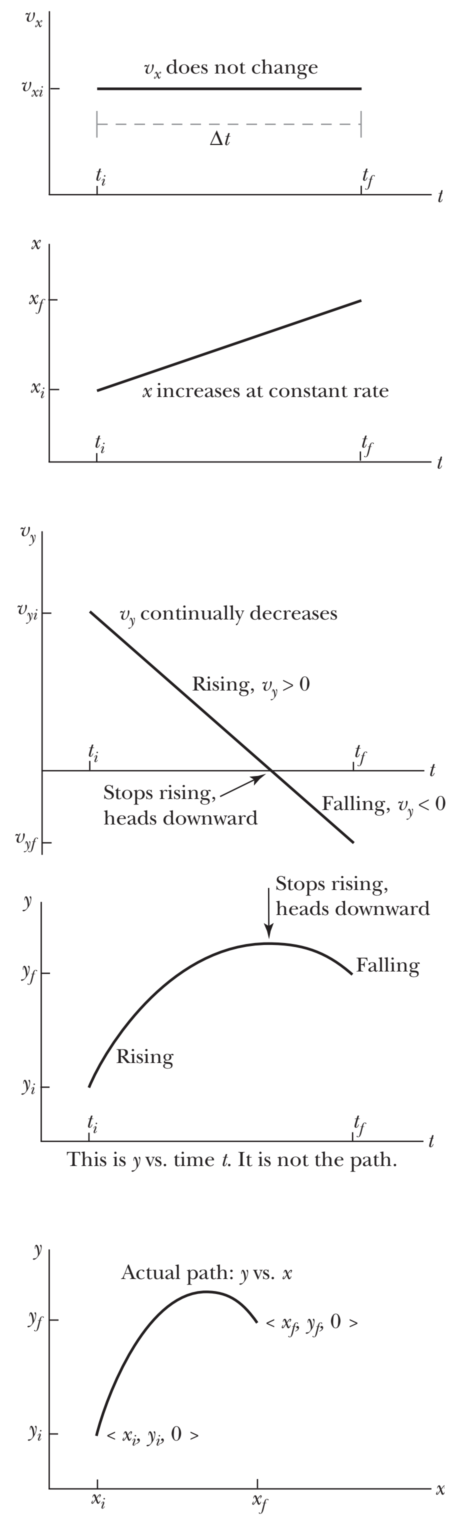


Figure 2.16 Graphs for the thrown ball.

Understanding the results

Let's explore the results a bit to understand what they tell us about the motion. The x component of the motion is very simple; it is completely unaffected by the downward-pointing gravitational force, so x simply increases at a constant rate (v_x constant). Of course, once the ball hits the ground, x no longer increases.

The y component of the motion is more interesting. The equation

$$v_{yf} = v_{yi} - g\Delta t$$

says that if you throw the ball so that v_{yi} is positive (heading up), at some time v_y will decrease to zero, and then become negative.

? At what time Δt_{top} will the ball reach its highest point?

Our result for y as a function of time:

$$y_f = y_i + v_{yi}\Delta t - \frac{1}{2}g(\Delta t)^2$$

does not help us answer this question, because we don't yet know how high the ball will go (we don't know what to use for y_f).

? Can we use the equation for v_y as a function of time to find the time Δt_{top} at which the ball reaches its highest point? Do we know the value of v_y when the ball reaches its highest point?

Just before the ball reaches its maximum height, v_y is positive. Just after it reaches its maximum height and begins to head downward, v_y is negative. At the instant the ball reaches its maximum height, $v_y = 0$. Using this information, we can solve for the elapsed time at that instant:

$$\text{If } v_{yf} = v_{yi} - g\Delta t_{\text{top}} = 0 \text{ then } \Delta t_{\text{top}} = \frac{v_{yi}}{g}$$

? How high is the ball when it turns around and heads downward?

Now that we have a value for Δt_{top}, we can use that in our general result for the final position:

$$y_f = y_i + v_{yi}\Delta t_{\text{top}} - \frac{1}{2}g(\Delta t_{\text{top}})^2 = y_i + \left[v_{yi}\left(\frac{v_{yi}}{g}\right) - \frac{1}{2}g\left(\frac{v_{yi}}{g}\right)^2\right]$$

$$y_f - y_i = \frac{v_{yi}^2}{2g} \text{ is the maximum height above your hand}$$

Check the algebra yourself to verify this result. This result makes sense, in that the bigger the initial y component of velocity, the higher the ball will go. Notice that it doesn't matter at all what x component of initial velocity you give the ball; all that matters is v_{yi}. However, it takes more effort to give the ball some v_{xi} in addition to giving it some v_{yi}, so for maximum height you want to throw the ball nearly straight up.

? Suppose you throw the ball so that it rises 2 m before falling back down. If you double v_{yi}, how far will it rise above your hand?

Since $y_f - y_i = v_{yi}^2/(2g)$, doubling v_{yi} will make the ball go up 4 times as high, to 8 m above your hand.

? Suppose you throw the ball so that it rises 2 m on Earth. If you give the ball the same v_{yi} on the Moon, where g is about one-sixth that on Earth, how far will the ball rise above your hand?

Since $y_f - y_i = v_{yi}^2/(2g)$, if g decreases by a factor of 6 but v_{yi} stays the same, the rise of the ball will increase by a factor of 6, so the ball will rise to 12 m above your hand. Because there is no air on the Moon, our analysis works well there.

? On Earth, what v_{yi} must you give the ball so that it rises 10 m above your hand?

We have $10\text{ m} = v_{yi}^2/(2(9.8\text{ N/kg}))$, and solving this we find $v_{yi} = 14\text{ m/s}$. This is a high enough speed that air resistance might be significant, and our result inaccurate.

? Suppose the ball is caught by a friend at the same height as you launched the ball. What is the formula for how long the ball was in the air?

In this case we have $y_f = y_i$, so $y_f = y_i + (v_{yi} - \frac{1}{2}g\Delta t)\Delta t$ gives us

$$\Delta t_{\text{up and down}} = 2\frac{v_{yi}}{g}\text{; compare with } \Delta t_{\text{top}} = \frac{v_{yi}}{g}$$

The ball takes twice as long to go up and down as to go up, so we conclude that the time to come down is the same as the time to go up. (In the presence of air resistance, the speed at every height is slower on the way down, so it takes longer to come down than to go up.) Again, notice that the time to go up and down doesn't depend at all on the x component of velocity. The x and y motions are independent of each other.

? How far from you is your friend when the ball is caught? Note that you know how long the ball was in the air, and you know the x component of velocity.

$$x_f = x_i + v_{xi}\Delta t_{\text{up and down}}\text{, so the distance is } x_f - x_i = v_{xi}\left(2\frac{v_{yi}}{g}\right)$$

This distance is called the "range" of the ball. It is the distance the ball goes in the air if it returns to the same height. It depends on v_y because that determines how much time it spends in the air. The result also depends on v_x because that determines how far the ball will move in the x direction while the ball is in the air.

Initial speed and angle

Sometimes you know the initial speed $|\vec{v}|$ and angle θ for the launch of a ball. Figure 2.17 shows how to calculate the sine and cosine of the angle. Use these results to solve for the velocity components:

$$v_{xi} = |\vec{v}|\cos\theta,\ v_{yi} = |\vec{v}|\sin\theta$$

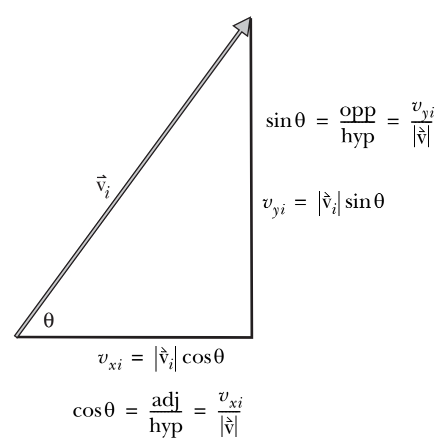


Figure 2.17 Converting speed and angle to velocity components.

Ex. 2.14 A ball is kicked from a location $\langle 9, 0, -5\rangle$ m (on the ground) with initial velocity $\langle -10, 13, -5\rangle$ m/s.

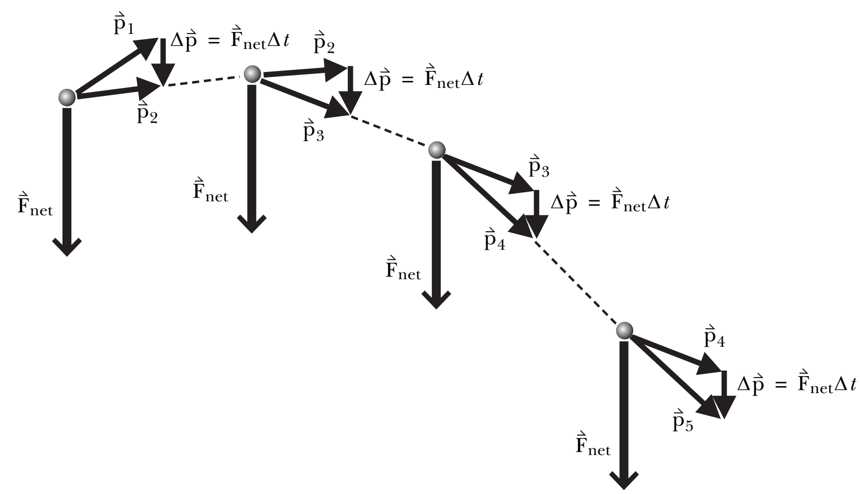
(a) What is the velocity of the ball 0.6 seconds after being thrown?
(b) What is the location of the ball 0.6 seconds after being thrown?
(c) What is the maximum height reached by the ball?
(d) At what time does the ball reach its maximum height?
(e) At what time does the ball hit the ground?
(f) What is the location of the ball when it hits the ground?

Ex. 2.15 Apply the general results obtained in the full analysis on page 64 to answer the following questions. You hold a small metal ball of mass m a height h above the floor. You let go, and the ball falls to the floor. Choose the origin of the coordinate system to be on the floor where the ball hits, with y up as usual. Just after release, what are y_i and v_{yi}? Just before hitting the floor, what is y_f? How much time Δt does it take for the ball to fall? What is v_{yf} just before hitting the floor? Express all results in terms of *m*, *g*, and *h*. How would your results change if the ball had twice the mass?

Ex. 2.16 A soccer ball is kicked at an angle of 60° to the horizontal with an initial speed of 10 m/s. Assume that we can neglect air resistance. For how much time is the ball in the air? How far does it go (horizontal distance along the field)? How high does it go?

2.4.4 Graphical prediction of motion

It is instructive to apply the Momentum Principle qualitatively and graphically to the thrown ball, to see visually how the Momentum Principle determines the motion. The Momentum Principle predicts that in a short time interval Δt, the momentum of the ball will change by an amount $\Delta\vec{p} = \vec{F}_{net}\Delta t$ which we can calculate, because we know the force acting on the ball. In the following diagram we show the initial momentum $\vec{p}_1$, the change in the momentum $\Delta\vec{p} = \vec{F}_{net}\Delta t$, and the new momentum $\vec{p}_2 = \vec{p}_1 + \Delta\vec{p}$. Note that this vector addition corresponds to adding the arrows $\vec{p}_1$ and $\Delta\vec{p}$ tip to tail. We approximate the average momentum in the time interval Δt by the new momentum $\vec{p}_2$, and we advance the ball in the direction of $\vec{p}_2$.



Graphical prediction of the motion of a ball, for three successive time steps.

During the short time interval Δt, the ball will move an amount $\vec{v}\Delta t$, in the direction of its new velocity $\vec{v}_2 = \vec{p}_2/m$ (we assume that $v << c$). The velocity is changing during this time interval, but if Δt is quite small (as it could be if we let a computer do the calculations), the change in the velocity is quite small, and it doesn't matter very much whether we use the velocity at the start or end of the time interval, or some kind of average. In this particular case of constant net force the average velocity is the arithmetic average, but if we included air resistance the net force wouldn't be constant in magnitude or direction, yet this method would still be accurate as long as we use small Δt.

At the next position we repeat the procedure, graphically adding the vectors $\vec{p}_2$ and $\Delta\vec{p} = \vec{F}_{net}\Delta t$ to obtain the new momentum $\vec{p}_3$. We then advance the ball in the direction of $\vec{p}_3$. As we do this repeatedly, we trace out graphically the trajectory of the ball. You can see that the trajectory looks like what is observed in the real world, and you can even see the increasing magnitude of the ball as it falls, corresponding to increasing speed. The important point to see in the diagram is that the net impulse $\vec{F}_{net}\Delta t$ *changes* the momentum.

2.4.5 Example: Block on spring (1D, nonconstant net force)

Next we'll study how to predict motion when there is more than one force, and the net force isn't constant. The situation we'll analyze is the motion of a block bouncing on a vertical spring. The spring has a relaxed length of 20 cm (0.2 m) and the spring stiffness is 8 N/m (Figure 2.18). You place a 60 gram block (0.06 kg) on top of the spring, compressing the spring, and the block sits motionless on the spring (Figure 2.19).

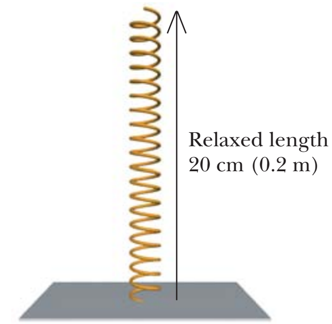


Figure 2.18 A relaxed vertical spring.

? How much is the spring compressed? Prove this using the Momentum Principle. Remember that $|\vec{F}_{\text{spring}}| = k_s|s|$, where s is the stretch of the spring (a negative number if the spring is compressed).

Since the block sits motionless over a time interval Δt, its momentum is not changing, which implies that the net force acting on the block must be zero (Figure 2.19):

$$\Delta\vec{p} = \vec{F}_{\text{net}}\Delta t$$
$$\langle 0, 0, 0\rangle = \langle 0, 0, 0\rangle(\Delta t)$$

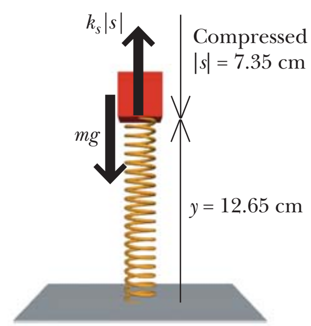


Figure 2.19 A block sits motionless on the spring. The net force must be zero.

The net force is the vector sum of all the individual forces acting on the block: the force due to the spring and the force due to the Earth.

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_{\text{spring}} + \vec{F}_{\text{Earth}}\\ &= \langle 0, k_s|s|, 0\rangle + \langle 0, -mg, 0\rangle\\ &= \langle 0, k_s|s| - mg, 0\rangle\end{aligned}$$

where k_s is the spring stiffness and as defined on page 54, $s = \Delta L = L - L_0$ is the compression of the spring (the current length minus the relaxed length, which is a negative number for a compression, so we've written $|s|$). Considering just the y component, we have

$$F_{net,\,y} = 0 = (8\text{ N/m})|s| - (0.06\text{kg})(9.8\text{ N/kg})$$

$$|s| = \frac{(0.06\text{kg})(9.8\text{ N/kg})}{(8\text{ N/m})} = 0.0735\text{ m}\ \ (\text{or } 7.35\text{ cm})$$

We'll choose the origin of our coordinate system to be at the bottom of the spring, where it is attached to the floor. As usual, we take $+y$ to be up. When the block is sitting motionless on the spring, the bottom of the block (or top of the spring) is at $y = (0.2 - 0.0735)\text{ m} = 0.1265\text{ m}$.

Next you push the mass down, compressing the spring an additional amount. You release the block, and because the spring has been compressed an extra amount, the upward force of the spring is greater than the downward gravitational force exerted by the Earth, and the block will move upward with increasing momentum due to the upward net force (Figure 2.20). After release, as the block passes a point 8 cm (0.08 m) above the floor, the block has an upward speed of 0.3 m/s.

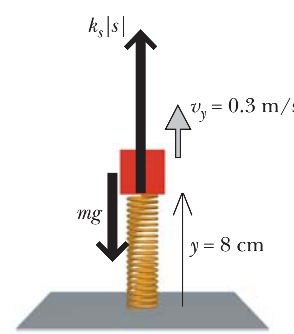


Figure 2.20 You compressed the spring, then released; the block heads upward with increasing speed, because the net force is nonzero and upward.

We will predict the velocity and position of the block after 0.02 s:

1. Choose a system

System: the block

2. List external objects that interact with the system, with diagram

the Earth, the spring, the air
(see force diagram on previous page)

We make the approximation that air resistance is negligible compared to the other forces.

3. Apply the Momentum Principle

$$p_{yf} = p_{yi} + F_{\text{net},y}\Delta t \text{ with } v << c \text{ so that } p_y \approx mv_y \text{ and } v_y \approx \frac{p_y}{m}$$

Initially the spring is compressed $(0.2 - 0.08)\text{m} = 0.12 \text{ m}$

$$F_{\text{net},y} = k_s|s| - mg = (8 \text{ N/kg})(0.12 \text{ m}) - (0.06 \text{ kg})(9.8 \text{ N/kg})$$

$$F_{\text{net},y} = 0.372 \text{ N}$$

$$p_{yf} = (0.06 \text{ kg})(0.3 \text{ m/s}) + (0.372 \text{ N})(0.02 \text{ s})$$

$$p_{yf} = 0.02544 \text{ kg·m/s}\text{ ; the new } p_y$$

$$v_{yf} \approx \frac{p_{yf}}{m} = \frac{(0.02544 \text{ kg·m/s})}{(0.06 \text{ kg})} = 0.424 \text{ m/s}\text{ ; the new } v_y$$

The net force is changing as the compression changes, so the momentum update is approximate.

4. Apply the position update formula

$$\vec{r}_f = \vec{r}_i + \vec{v}_{\text{arg}}\Delta t\text{, or } y_f = y_i + v_{\text{avg},y}\Delta t$$

Make the approximation that the net force was nearly constant during the 0.02 s:

$$v_{\text{avg},y} = \frac{(v_{yi} + v_{yf})}{2} = \frac{(0.3 + 0.424) \text{ m/s}}{2} = 0.362 \text{ m/s}$$

$$y_f = (0.08 \text{ m}) + (0.362 \text{ m/s})(0.02 \text{ s}) = 0.08724 \text{ m}\text{ ; the new } y$$

5. There are no remaining unknowns

6. Check

The units check for the y_f; both v_y and y have increased as expected.

During the 0.02 s time interval the block is moving up and the length of the spring is changing, so the stretch (compression) of the spring is changing. That means that the spring force $k_s|s|$ is not constant during the 0.02 s. Therefore when we use $p_{yf} = p_{yi} + F_{\text{net},y}\Delta t$ to update the momentum, we're making an approximation that the force doesn't change very much during that time.

We can estimate how serious this issue is by calculating the net force at the new position, $y_f = 0.08724 \text{ m}$.

? What is the new net force? (You need to determine the new compression.)

With the top of the spring now at $y_f = 0.08724 \text{ m}$, the compression of the spring is $|s| = (0.2 - 0.08724) \text{ m} = 0.11276 \text{ m}$, and the net force has become this:

$$F_{\text{net},y} = k_s|s| - mg = (8 \text{ N/kg})(0.11276 \text{ m}) - (0.06 \text{ kg})(9.8 \text{ N/kg})$$

$$F_{\text{net},y} = 0.314 \text{ N}$$

The net force at the start of the time interval was $F_{\text{net},y} = 0.372\text{ N}$. So although the net force did change during the 0.02 s, it didn't change very much, and our approximate analysis is therefore pretty good.

? How could our predictions be improved?

By taking shorter time steps. Instead of taking one step of 0.02 s, we could take two steps of 0.01 s, or ten steps of 0.002 s. During each of these shorter time intervals the compression would change less, so the force would be more nearly constant. Unfortunately, to achieve increased accuracy we have to do a lot more calculations.

Taking another step

To continue predicting the motion of the block into the future, we can take another step. The final values of momentum and position after the original 0.02 s step become the initial values for the next 0.02 s step:

The initial momentum and position are taken from the final momentum and position in the previous step.

3. Apply the Momentum Principle

$$p_{yf} = p_{yi} + F_{\text{net},y}\Delta t \text{ with } v << c \text{ so that } p_y \approx mv_y \text{ and } v_y \approx \frac{p_y}{m}$$

Spring is compressed $(0.2 - 0.08724)\text{m} = 0.11276\text{ m}$

$$F_{\text{net},y} = k_s|s| - mg$$
$$= (8\text{ N/kg})(0.11276\text{ m}) - (0.06\text{ kg})(9.8\text{ N/kg})$$

$$F_{\text{net},y} = 0.314\text{ N}$$

$$p_{yf} = (0.02544\text{ kg·m/s}) + (0.314\text{ N})(0.02\text{ s})$$

$$p_{yf} = 0.03172\text{ kg·m/s}\text{; the new } p_y$$

$$v_{yf} \approx \frac{p_{yf}}{m} = \frac{(0.03172\text{ kg·m/s})}{(0.06\text{ kg})} = 0.5287\text{ m/s}\text{; the new } v_y$$

4. Apply the position update formula

$$\vec{r}_f = \vec{r}_i + \vec{v}_{\text{arg}}\Delta t\text{, or } y_f = y_i + v_{\text{avg},y}\Delta t$$

Make the approximation that the net force was nearly constant during the 0.02 s:

$$v_{\text{avg},y} = \frac{(v_{yi} + v_{yf})}{2} = \frac{(0.424 + 0.5287)\text{ m/s}}{2} = 0.4764\text{ m/s}$$

$$y_f = (0.08724\text{ m}) + (0.4764\text{ m/s})(0.02\text{ s}) = 0.09677\text{ m}$$

Look through this calculation and make sure you understand how the *final* results for position and momentum from the first 0.02 s time step have been used as the *initial* values for the second 0.02 s time step.

In principle we could continue, taking more and more steps to predict farther and farther into the future. Here is a summary of this iterative scheme:

- Start with initial positions and momenta of the interacting objects.
- Calculate the (vector) forces acting on each object.
- Update the momentum of each object: $\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t$.
- Update the positions: $\vec{r}_f = \vec{r}_i + \vec{v}_{\text{arg}}\Delta t$.
- Repeat.

Every time you repeat, the "final" momentum and position become the "initial" momentum and position for the next step. You have to use an approximate value for $\vec{v}_{avg}$, either by using the velocity at the start or end of the time interval, or by taking the arithmetic average of these two velocities as we did above.

While this scheme is very general, doing it by hand is incredibly tedious. It is possible to program a computer to do these calculations repetitively. Computers are now fast enough that it is possible to get high accuracy simply by taking very short time steps, so that during each step the net force and velocity aren't changing much. We'll talk more about computer prediction of motion later in this chapter.

Ex. 2.17 After a third time step of 0.02 seconds, what will be the position and momentum of the block?

2.4.6 Example: Fast proton (1D, constant net force, relativistic)

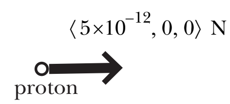
A proton in a particle accelerator is moving with velocity $\langle 0.96c, 0, 0\rangle$, so the speed is $0.96 \times 3\times10^8\ \text{m/s} = 2.88\times10^8\ \text{m/s}$. A constant electric force is applied to the proton to speed it up, $\vec{F}_{net} = \langle 5\times10^{-12}, 0, 0\rangle\ \text{N}$. What is the proton's speed as a fraction of the speed of light after 20 nanoseconds ($1\ \text{ns} = 1\times10^{-9}\ \text{s}$)?

1. Choose a system

System: the proton

2. List external objects that interact with the system, with diagram

electric charges in the accelerator
(a circle represents the system)



3. Apply the Momentum Principle

$$\vec{p}_f = \vec{p}_i + \vec{F}_{net}\Delta t$$

$$\langle p_{xf}, 0, 0\rangle = \langle \gamma_i m v_{xi}, 0, 0\rangle + (\langle 5\times10^{-12}, 0, 0\rangle\ \text{N})(20\times10^{-9}\ \text{s})$$

$$p_{xf} = \frac{1}{\sqrt{1-\left(\frac{0.96c}{c}\right)^2}}(1.7\times10^{-27}\ \text{kg})(0.96\times3\times10^8\ \text{m/s}) + (1\times10^{-19}\ \text{N·s})$$

$$p_{xf} = (1.75\times10^{-18}\ \text{kg·m/s}) + (1\times10^{-19}\ \text{N·s}) = 1.85\times10^{-18}\ \text{kg·m/s}$$

$$\frac{v_{xf}}{c} = \frac{\frac{p_{xf}}{mc}}{\sqrt{1+\left(\frac{p_{xf}}{mc}\right)^2}}$$

(See Section 2.15, page 98; obtaining v from p when the speed is near the speed of light.)

Evaluate $\frac{p_{xf}}{mc} = \frac{1.85\times10^{-18}\ \text{kg·m/s}}{(1.7\times10^{-27}\ \text{kg})(3\times10^8\ \text{m/s})} = 3.62$ (no units)

$$\frac{v_{xf}}{c} = \frac{3.62}{\sqrt{1+3.62^2}} = 0.964$$

The speed didn't increase very much, because the proton's initial speed, $0.96c$, was already close to the cosmic speed limit, c. Because the speed hardly changed, the distance the proton moved during the 20 ns was approximately equal to $(0.96\times3\times10^8\ \text{m/s})(20\times10^{-9}\ \text{s}) = 5.8\ \text{m}$.

2.5 Problems of greater complexity

So far all of the examples we have considered have involved finding a change in momentum (and position), given a known force acting over a known time interval. The following problems require you to find either the duration of an interaction (time interval), or the force exerted during an interaction. These large problems involve several steps in reasoning.

2.5.1 Example: Strike a hockey puck

In Figure 2.21 an 0.4 kg hockey puck is sliding along the ice with velocity $\langle 20, 0, 0\rangle$ m/s As the puck slides past location $\langle 1, 0, 2\rangle$ m on the rink, a player strikes the puck with a sudden force in the $+z$ direction, and the hockey stick breaks. Some time later, the puck's position on the rink is $\langle 13, 0, 21\rangle$ m . When we pile weights on the side of a hockey stick we find that the stick breaks under a force of about 1000 N (this is roughly 250 pounds; a force of one newton is equivalent to a force of about a quarter of a pound, approximately the weight of a small apple).

(a) For approximately how much time $\Delta t_{\text{contact}}$ was the hockey stick in contact with the puck? Evidently the contact time is quite short, since you hear a short, sharp crack. Be sure to show clearly the steps in your analysis.

(b) What approximations and/or simplifying assumptions did you make in your analysis?

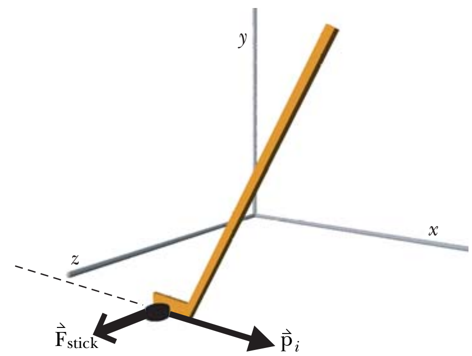


Figure 2.21 A hockey stick hits a puck as it slides by.

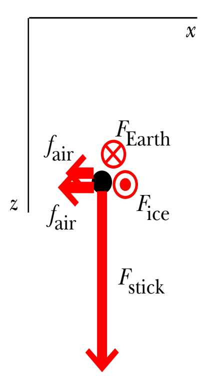
1. Choose a system

System: the hockey puck

2. List external objects that interact with the system, with diagram

Earth, ice, hockey stick, air

List all interacting objects, even if you think some interactions will cancel out.



We're looking down on the ice from above. The y axis points out of the page, toward you.

The symbol ⊙ means out of the page (the tip of an arrow pointing out at you); the symbol ⊗ means into the page (tail feathers of an arrow pointing away from you)

When possible, make a 2D diagram, which is much easier than trying to draw in 3D.

3. Apply the Momentum Principle

$$\langle p_{xf}, 0, p_{zf}\rangle = \langle p_{xi}, 0, 0\rangle + \langle -f_{\text{ice}} - f_{\text{air}}, (F_{\text{ice}} - mg), F_{\text{stick}}\rangle \Delta t_{\text{contact}}$$

We use the information that the puck slides without bouncing on the ice, which means that p_y is zero at all times.

Write the three component equations separately:

x component: $p_{xf} = p_{xi} - (f_{\text{ice}} + f_{\text{air}})\Delta t_{\text{contact}}$

y component: $0 = (F_{\text{ice}} - mg)\Delta t_{\text{contact}}$ which implies $F_{\text{ice}} = mg$

z component: $p_{zf} = F_{\text{stick}}\Delta t_{\text{contact}}$

The atoms in the top layer of the ice exert upward forces on the atoms on the bottom of the puck, counteracting the downward gravitational pull of the Earth.

Find new momentum:

$f_{\text{ice}} \approx 0$ and $f_{\text{air}} \approx 0$ (negligible forces along x direction)

$p_{xf} \approx p_{xi} + (0)\Delta t_{\text{contact}} \approx p_{xi}$ so no change in p_x

$p_{zf} = F_{\text{stick}}\Delta t_{\text{contact}}$

Since the puck slides long distances with little change in velocity, friction and air resistance must be negligible.

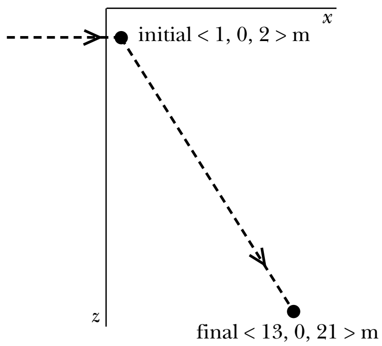


Figure 2.22 The x component of the momentum (and velocity) hardly changes, but the z component of momentum (and velocity) changes quickly from zero to some final value when the puck is hit.

? Given these results from the Momentum Principle for p_x and p_z, make a sketch of the path of the puck before and after it is hit.

The path of the puck must look something like that shown in Figure 2.22.

In the result $p_{zf} = F_{\text{stick}}\Delta t_{\text{contact}}$ there are two unknowns, p_{zf} and F_{stick}. We need another equation in order to be able to solve for the unknown contact time $\Delta t_{\text{contact}}$. We can get additional information from the position update formula $\vec{r}_f = \vec{r}_i + \vec{v}_{\text{avg}}\Delta t$.

4. Apply the position update formula

In $\vec{r}_f = \vec{r}_i + \vec{v}_{\text{arg}}\Delta t$, let Δt_{slide} be the amount of time it takes the puck to slide from where it was struck to the known final position:

$$\langle 13, 0, 21\rangle \text{ m} = (\langle 1, 0, 2\rangle \text{ m}) + \langle 20 \text{ m/s}, 0, v_z\rangle \Delta t_{\text{slide}}$$

x component: $(13 \text{ m}) = (1 \text{ m}) + (20 \text{ m/s})\Delta t_{\text{slide}}$, so

$$\Delta t_{\text{slide}} = (12 \text{ m})/(20 \text{ m/s}) = 0.6 \text{ s}$$

y component: $0 = 0 + 0\Delta t_{\text{slide}}$ so $0 = 0$ (not a surprise)

z component: $(21 \text{ m}) = (2 \text{ m}) + v_z(0.6 \text{ s})$ since $\Delta t_{\text{slide}} = 0.6 \text{ s}$, so

$$v_z = (19 \text{ m})/(0.6 \text{ s}) = 31.7 \text{ m/s}$$

Since the puck slides long distances with little change in velocity, friction and air resistance must be negligible. We assume that after the puck is struck, the velocity is nearly constant (with a new magnitude and direction). Therefore the average velocity is the same as the velocity just after impact.

5. Solve for the unknowns

$$p_{zf} = F_{\text{stick}}\Delta t_{\text{contact}} \text{ where } p_{zf} \approx m v_{zf} \text{ (since } v << c)$$

$$(0.4 \text{ kg})(31.7 \text{ m/s}) = (1000 \text{ N})\Delta t_{\text{contact}}$$

$$\Delta t_{\text{contact}} = (0.4 \text{ kg})(31.7 \text{ m/s})/(1000 \text{ N}) = 0.013 \text{ s}$$

The contact time is very short, a small fraction of a second. As a result, you hear a short sharp crack when the stick hits the puck.

6. Check

- Units check (contact time is in seconds)
- Is the result reasonable? The contact time is very short, as expected. If for example it had come out to be 300 s (5 minutes!) we should check our calculations.

How good were our approximations?

We made the following approximations and simplifying assumptions:
- Ice exerts little force in the x or z directions; low sliding friction.
- Negligible air resistance.
- Force of stick is roughly constant during $\Delta t_{\text{contact}}$ and equal to 1000 N.
- The puck doesn't move very far during the contact time.

The neglect of sliding friction and air resistance is probably pretty good, since a hockey puck slides for long distances on ice.

We know the hockey stick exerts a maximum force of $F_{\text{stick}} = 1000$ N, because we observe that the stick breaks. We approximate the force as nearly constant during contact. Actually, this force grows quickly from zero at first contact to 1000 N, then abruptly drops to zero when the stick breaks.

The final approximation is somewhat questionable. Although 0.013 s is a short time, the puck moves (20 m/s)(0.013 s) = 0.26 m (a bit less than one foot) in the x direction during this time. Also during this time v_z increases from 0 to 31.7 m/s, with an average value of about 15.8 m/s, so the z displacement is about (15.8 m/s)(0.013 s) = 0.2 m during contact. On the other hand, these displacements aren't very large compared to the displacement from $\langle 1, 0, 2\rangle$ m to $\langle 13, 0, 21\rangle$ m, so our result isn't terribly inaccurate due to this approximation. Nevertheless, a more accurate sketch of the path of the puck should show a bend as in Figure 2.23.

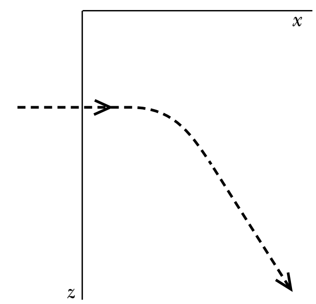


Figure 2.23 A more accurate overhead view of the path of the hockey puck, showing the bend during impact.

We can even calculate the radius of curvature of this bend, the radius of the "kissing circle" discussed in the previous chapter. We know that

$$\frac{d\vec{p}}{dt} = \vec{F}_{\text{net}}$$

$$\text{On the left, } \left|\frac{d\vec{p}}{dt}\right| = \frac{|\vec{v}|}{R}|\vec{p}| \approx \frac{m|\vec{v}|^2}{R} = \frac{(0.4\text{ kg})(20\text{ m/s})^2}{R} \text{ since } \vec{p} \approx m\vec{v}$$

At first contact the velocity is in the x direction and has magnitude of 20 m/s. The kissing circle is tangent to the incoming velocity.

$$\text{On the right, } |\vec{F}_{\text{net}}| = 1000\text{ N}$$

$$\text{Equate the left and right quantities: } \frac{(0.4\text{ kg})(20\text{ m/s})^2}{R} = 1000\text{ N}$$

$$\text{Solve: } R = \frac{(0.4\text{ kg})(20\text{ m/s})^2}{(1000\text{ N})} = 0.16\text{ m}$$

This is a plausible result for the radius of the bend, since we saw that x changes by about 0.26 m and z changes by about 0.2 m during contact.

It is important to see that even though our analysis of the stick contact time (0.013 s) isn't exact, it is adequate to get a reasonably good determination of this short time, something that we wouldn't know without using the Momentum Principle and the position update formula. The short duration of the impact explains why we hear a sharp, short crack.

Choice of system

? We chose the hockey puck as the system to analyze. Why not choose the system consisting of both the hockey puck and the hockey stick?

The problem with choosing both objects as the system is that the 1000 N force is now an internal force and doesn't show up in the Momentum Principle, so we aren't able to use this information. By the reciprocity of electric forces, including the interatomic forces between stick and puck, the puck exerts a 1000 N force on the stick. In the combined system the puck gains momentum from the stick, and the stick loses momentum to the puck. The total momentum of the system doesn't change: $\vec{p}_{\text{system}} = \vec{0}$.

Review

Let's review what we did to analyze this situation, in the form of a general scheme for attacking problems. We can summarize our work with a diagram in the shape of a diamond, which emphasizes

- starting from the Momentum Principle applied to a system,
- then expanding for the particular situation,
- then contracting down to solving for the quantity of interest,
- followed by checking for reasonableness.

Link the lines in this diamond to the hockey stick analysis:

1. Choose a system.
2. List interacting objects, with diagram.
3. Apply the Momentum Principle:

$$\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t$$

Expand the Momentum Principle by substituting any values you know.

4. You may also need to expand $\vec{r}_f = \vec{r}_i + \vec{v}_{\text{arg}}\Delta t$.
5. Solve for the unknowns.
6. Check.

An important point implied by this diamond is that it is usually not useful to try to jump immediately to the step "Solve for the unknowns," by hunting for some formula that gives the unknown quantity directly. Very often such a special-case formula may not exist. For example, in the hockey puck problem the contact time emerged from applying the Momentum Principle and the position update formula. There wasn't some ready-made "formula for the contact time for hockey sticks and pucks" that you could use.

If you start by applying the Momentum Principle to a chosen system you can attack novel problems that you've never before encountered. The Momentum Principle is always valid, whereas special-case formulas aren't.

In previous studies you may have been taught a useful but restricted approach to solving problems, which is to start with a formula for the unknown quantity. That is, if you've trying to find v, start with a formula for v. We will help you learn a more powerful technique for solving problems, which is to start from a fundamental physics principle (in this case the Momentum Principle), expand it by substituting known values, then contract to solve for the unknown quantities. In other words, you *derive* the formula you need rather than hunt for it. This is the only technique that can give you the power to solve novel problems, ones that no one has previously encountered.

In an increasingly complex world, engineers and scientists are continually being asked to meet new challenges. A major goal of this course is to prepare you to meet the novel challenges of the 21st century.

2.5.2 Example: Colliding students

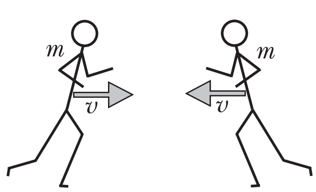


Figure 2.24 Colliding students. The speed (magnitude of velocity) of each student is assumed to be the same and is represented by v. We also make the simplifying assumption that the students have about the same mass m.

Next we'll work through in detail a messy real-world situation: Two students who are late for tests are running to classes in opposite directions as fast as they can (Figure 2.24). They turn a corner, run into each other head-on, and crumple into a heap on the ground. Using physics principles, estimate the force that one student exerts on the other during the collision. You will need to estimate some quantities; give reasons for your choices and provide checks showing that your estimates are physically reasonable.

This problem is rather ill-defined and doesn't seem much like a "textbook" problem. No numbers have been given, yet you're asked to estimate the force of the collision. This kind of problem is typical of the kinds of problems engineers and scientists encounter in their professional work. For example, suppose you are trying to design a crash helmet and you need to estimate the force it must withstand without breaking. You don't know exactly what the ultimate wearer will be doing at the time of a crash, so you have to make some reasonable estimates of typical human activities on which to base your analysis. We'll make the simplifying assumption that the students have similar masses and similar speeds (Figure 2.24).

Remember the diamond scheme as a guide to how to proceed. We'll carry out the analysis symbolically and plug in estimated values of student mass and speed at the end. That way we get a general solution that can be evaluated for different values of these quantities.

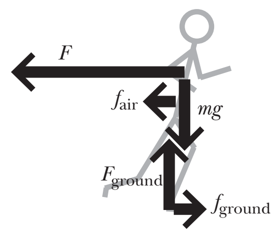
1. Choose a system

System: the student on the left

2. List external objects that interact with the system, with diagram

Earth, ground, other student, air

(more complex diagram shows point of application of each force)



3. Apply the Momentum Principle

$$\langle 0, 0, 0\rangle = \langle p_{xi}, 0, 0\rangle + \langle -F - f_{\text{ground}} - f_{\text{air}}, (F_{\text{ground}} - mg), 0\rangle\Delta t$$

Write the three component equations separately:

x component: $0 = p_{xi} - (F + f_{\text{ground}} + f_{\text{air}})\Delta t$

y component: $0 = (F_{\text{ground}} - mg)\Delta t$, so $F_{\text{ground}} = mg$

z component: $0 = 0$ which is true but uninformative

Find new momentum:

Assume f_{ground} and f_{air} are negligible compared to F

$$0 = p_{xi} - F\Delta t, \text{ so } mv = F\Delta t \text{ since } v << c$$

List all interacting objects, even if you think some interactions will cancel out.

The student's heels are pushed to the left along the ground due to the collision, and the resisting ground pushes to the right. We'll assume that this force is much smaller than the force exerted by the other student, because the student's shoes can slip.

Many of the momentum components are zero. Do you see why $p_{xf} = 0$?

In a vector equation, the x component on the left must equal the x component on the right; similarly for y and z.

We could estimate the typical mass of a student, and the likely speed of a student running at full speed, but $mv = F\Delta t$ is one equation in two unknowns, the unknown force F of the other student and the time duration Δt of the impact.

? Before trying to apply the position update formula $\vec{r}_f = \vec{r}_i + \vec{v}_{\text{arg}}\Delta t$, try to estimate Δt of the impact simply by making a guess.

It is common to guess what seems a rather short time, such as a second, or half a second. However, it is easy to show that such estimates are way off. The record for the 100 m dash is about 10 seconds. If we assume that these students are very fast runners, they each could be running 10 m/s. An alternative way of making the estimate is to note that you can easily walk about 4 miles in an hour, which gives

$$v \approx \left(4\frac{\text{mi}}{\text{hr}}\right)\left(1.6\frac{\text{km}}{\text{mi}}\right)\left(1000\frac{\text{m}}{\text{km}}\right)\left(\frac{1\text{ hr}}{3600\text{ s}}\right) \approx 1.8 \text{ m/s}$$

This moderate walking speed suggests that the students should be able to run between 5 m/s and 10 m/s. Let's estimate that they each have a speed $v \approx 5$ m/s. During the collision the speed drops quickly from v to 0, so the average speed during the collision is approximately $(v + 0)/2$, or 2.5 m/s.

? If we guess that Δt is about half a second, during that half second, how far does the student move?

At an average speed of 2.5 m/s, in a half second the student would go 1.25 meters, well over a yard. This would imply that the two students pass right through each other, coming out the other side! Evidently an estimate of a half second is wildly too big and is inconsistent with the real world.

The problem is that it is very difficult to guess short time intervals accurately. On the other hand, we can estimate short distances rather well, and this gives us an indirect way to estimate short time intervals. Each student gets squeezed during the collision. Suppose each student's body gets pushed in a distance of about $\Delta x \approx 5\text{ cm} = 0.05\text{ m}$ (try pushing on your stomach and see how much deflection you can make).

4. Apply the position update formula

Apply $\vec{r}_f = \vec{r}_i + \vec{v}_{avg}\Delta t$ or equivalently $\vec{v}_{avg} = \Delta\vec{r}/\Delta t$:

$$\langle v/2, 0, 0\rangle = \langle \Delta x, 0, 0\rangle/\Delta t$$

x component: $v/2 = \Delta x/\Delta t$, so $\Delta t = 2\Delta x/v$

5. Solve for the unknowns

Estimate that the student is brought to an abrupt stop in just 5 cm (about 2 inches).

$$mv = F\Delta t = F(2\Delta x/v)$$

$$F = \frac{mv^2}{2\Delta x} = \frac{(60\text{ kg})(5\text{ m/s})^2}{2(0.05\text{ m})} = 15000\text{ N}$$

$$\Delta t = \frac{2\Delta x}{v} = \frac{2(0.05\text{ m})}{5\text{ m/s}} = 0.02\text{ s}$$

Force has units (kg)(m/s)/s, which has the units of momentum (kg·m/s) divided by s, which is correct for a force (change of momentum divided by time).

The collision time is 25 times smaller than our original guess of 0.5 s. So the key to making a realistic estimate of a time interval is to estimate average speed and distance, and then derive a time estimate from that.

6. Check

- Units check (force is in newtons, collision time is in seconds)
- Is the result reasonable? The contact time is very short, as expected. Is the force reasonable or not? See discussion below.

15000 N is a very large force. For example, the gravitational force on a 60 kg student (the "weight") is only about $(60\text{ kg})(9.8\text{ N/kg}) \approx 600\text{ N}$. The force of the impact is more than 25 times the weight of the student! It's like having a stack of 25 students sitting on you. If the students hit heads instead of stomachs, the squeeze might be less than 1 cm, and the force would be over 5 times as large! This is why heads can break in such a collision.

So yes, our result of 15000 N is plausible: collisions involve very large forces acting for very short times, giving impulses of ordinary magnitude.

How good were our approximations?

We made the following approximations and simplifying assumptions:

- We estimated the running speed from known 100 m dash records.
- We estimated the masses of the students; we could plug in other masses now that we have derived general results in terms of m and v.
- We assumed that the horizontal component of the force of the ground on the bottom of the student's shoe was small compared to the force exerted by the other student. Now that we find that the impact force is huge, this assumption seems quite good.
- We made the approximation that the impact force was nearly constant during the impact, so what we've really determined is an average force.
- We assumed that the students had similar masses and similar running speeds, to simplify the analysis. If this is not the case, the analysis is significantly more complicated, but we would still find that the impact force is huge.

You might object that with all these estimates and simplifying assumptions the final result of 15000 N for the impact force is fatally flawed. It is certainly the case that we don't have a very accurate result. But nevertheless we gained valuable information, that the impact force is *very* large. Before doing this analysis based on the Momentum Principle, we had no idea of whether the force was small compared to the student's weight, comparable, or much bigger. Now we have a quantitative result that the force is on the order of 25 times the weight of a student, and we can appreciate why collisions are so dangerous.

? What if we choose the system to include both students?

In this case there is almost no external impulse during the brief time of contact. The big internal forces the students exert on each other are equal in magnitude but opposite in direction (due to the reciprocity of interatomic forces that come into play when the students make contact). Therefore these forces don't change the momentum of the combined system. Before the collision the total momentum of the combined system is zero ($mv - mv = 0$), and after the collision the total momentum is also zero (they're not moving). So there's no change in momentum, which is consistent with there being negligible external forces. Simple analysis, but with this choice of system we learn nothing about the force that one student exerts on the other. That's why we choose just one student as the system.

2.5.3 Physical models

Our model of the colliding student situation is good enough for many purposes. However, we left out some aspects of the actual motion. For example, we mostly ignored the flexible structure of the students, how much their shoes slip on the ground during the collision, etc. We have also quite sensibly neglected the gravitational force of Mars on the students, because it is so tiny compared to the force of one student on the other.

Making and using models is an activity central to physics, and the criteria for a "good" physical model will depend on how we intend to use the model. In fact, one of the most important problems a scientist or engineer faces is deciding what interactions must be included in a model of a real physical, chemical, or biological system, and what interactions can reasonably be ignored.

Simplified (or "idealized") models

If we neglect some (hopefully small) effects, we say that we are constructing a simplified model of the situation. A useful model should omit extraneous detail but retain the main features of the real-world situation. We hope that the main results of our analysis of the simplified model will apply adequately to the complex real-world situation.

Of course if we do a poor job of modeling and make inappropriate approximations or neglect effects that are actually sizable, we may get a rather inaccurate result (though perhaps adequate for some purposes). There is therefore a certain art to making a good model. One goal of this course is to help you develop skill in formulating simplified but meaningful physical models of complex situations.

Another common way of describing a model is to say that it is an "idealization," by which we mean a simple, clean, stripped-down representation, free of messy complexities. "Ideally," a ball will roll forever on a level floor, but a real ball rolling on a real floor eventually comes to a stop. An "ideal" gas is a fictitious gas in which the molecules don't interact with each other, as opposed to a real gas whose molecules do interact, but only when they come close to each other.

The analysis of a simplified model gives an approximate result that differs somewhat from what actually happens in the real world. Alternatively, you can consider your result to be exact for a *different* real-world situation, one where the neglected influences are not actually present. For example, the calculations we made concerning the colliding students could apply almost exactly to two rubber balls moving in outer space that compress by the amount we estimated.

An important aspect of physical modeling that will engage us throughout this course is making appropriate approximations to simplify the messy, real-world situation enough to permit (approximate) analysis using Newton's laws. Actually, using Newton's laws is itself an example of modeling and making approximations, because we are neglecting the effects of quantum mechanics and of general relativity (Einstein's treatment of gravitation). Newton's laws are only an approximation to the way the world works, though frequently an extremely good one.

2.6 Fundamental forces

There are four different kinds of fundamental forces currently known to science, associated with four different kinds of interactions: gravitational, electromagnetic, nuclear (also referred to as the "strong" interaction), and the "weak" interaction.

- The *gravitational* interaction is responsible for an attraction every object exerts on every other object. For example, the Earth exerts a gravitational force on the Moon, and the Moon exerts a gravitational force on the Earth.
- The *electromagnetic* interaction includes electric forces responsible for sparks, static cling, and the behavior of electronic circuits, and magnetic forces responsible for the operation of motors driven by electric current. Protons repel each other electrically, as do electrons, whereas protons and electrons attract each other (Figure 2.25). Electric forces bind protons and electrons to each other in atoms, and are responsible for the chemical bonds between atoms in molecules. The force of a stretched or compressed spring is due to electric forces between the atoms that make up the spring.
- The nuclear or *strong* interaction holds protons and neutrons together in the nucleus of an atom despite the large mutual electric repulsion of the protons (Figure 2.26). (The neutrons are not electrically charged and interact only through the strong force.)
- An example of the *weak* interaction is seen in the instability of a neutron. If a neutron is removed from a nucleus, with an average lifetime of about 15 minutes the neutron decays into a proton, an electron, and a ghostly particle called the antineutrino. This change is brought about by the weak interaction.

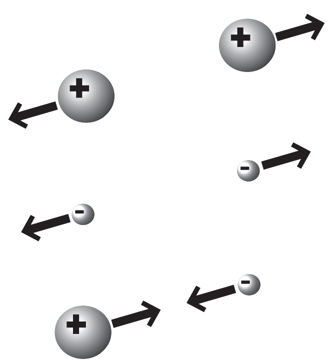


Figure 2.25 The electric force: protons repel each other; electrons repel each other; protons and electrons attract each other.

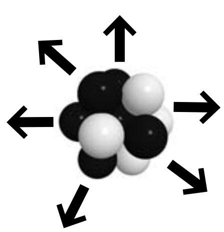


Figure 2.26 The strong force: the protons in the nucleus of an atom exert repulsive electric forces on each other, but he strong interaction holds the nucleus together despite this electric repulsion.

We will be mainly concerned with gravitational and electric interactions, but we will occasionally encounter situations where the nuclear or strong interaction plays an important role. We will have little to say about the weak interaction. Nor will we deal with magnetism, the other part of the electromagnetic interaction. The second volume of this textbook deals extensively with both electric and magnetic interactions.

2.7 The gravitational force law

To predict the motion of stars, planets, spacecraft, comets, satellites, and other massive objects traveling through space, we need to use a gravitational force law together with the Momentum Principle. As a matter of general culture, it can be interesting to understand how the motion of objects in space can be predicted by physics principles. If you are studying aerospace engineering or astrophysics this may also be a professional interest.

What if your interests are in nanotechnology or chemistry or civil engineering? Studying how to predict the motion of stars and planets is one of the most direct ways to understand in general how the Momentum Principle determines the behavior of objects in the real world. The motion of stars and planets is in important ways simpler than other mechanical phenomena, because there is no friction to worry about, so this is a good place to start your study. The basic ideas used to predict the motion of stars and planets can be applied later to a wide range of everyday and atomic phenomena.

In the 1600's Isaac Newton deduced that there must be an attractive force associated with a gravitational interaction between any pair of objects. The gravitational force acts along a line connecting the two objects (Figure 2.27), is proportional to the mass of one object and to the mass of the other object, and is inversely proportional to the square of the distance between the *centers* of the two objects (not the gap between their surfaces). Here is the formula for the gravitational force exerted on object 2 by object 1:

THE GRAVITATIONAL FORCE LAW

$$\vec{F}_{grav\text{ on 2 by 1}} = -G\frac{m_2 m_1}{|\vec{r}_{2\text{-}1}|^2}\hat{r}_{2\text{-}1}$$

$\vec{r}_{2\text{-}1}$ points from the center of object 1 to the center of object 2

G is a universal constant: $G = 6.7\times10^{-11}\ \dfrac{\mathrm{N\cdot m^2}}{\mathrm{kg^2}}$

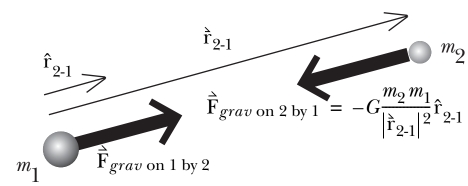


Figure 2.27 The gravitational force exerted on object 2 by object 1. (The force exerted on object 1 by object 2 has the same magnitude but opposite direction.)

This force law looks pretty complicated, but it could have been a lot more complicated than it is. For example, the gravitational force law does not depend on the momentum or velocity of the objects. It depends only on the masses, and on the position of one object relative to the other object.

Large spheres

An optional section at the end of this chapter (page 96) shows that uniform-density spheres interact gravitationally as though all of their mass were concentrated at the center of the sphere, so the gravitational force law applies to large uniform-density spheres as long as you use the center-to-center distance. For example, the force that the Earth exerts on you can be calculated as though all the mass of the Earth were at its center, 6.4×10^{6} m away from where you are sitting.

Measurement of *G*

Another optional section at the end of this chapter (page 97) tells how Cavendish in the late 1700's measured G by observing the tiny gravitational forces two lead balls exerted on each other.

2.7.1 Understanding the gravitational force law

We will spend some time understanding how to calculate forces between objects using Newton's gravitational force law. Then we will use this force in the Momentum Principle to predict the future motion of objects.

The gravitational force law involves a lot of different symbols and may look pretty intimidating at first. Let's take the law apart and look at the individual pieces to try to make sense of the formula.

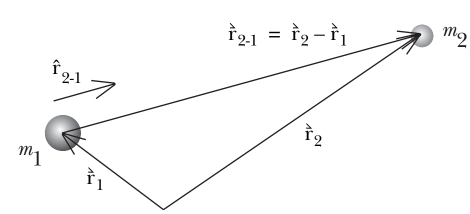


Figure 2.28 The location of object 2 relative to object 1: "final minus initial."

The relative position vector $\vec{r}_{2\text{-}1}$ in the gravitational force law points from the center of object 1 to the center of object 2, and so does the unit vector $\hat{r}_{2\text{-}1}$ ("r-hat") in Figure 2.28. Recall that in words $\vec{r}_{2\text{-}1}$ is the location of object 2 relative to object 1, "final minus initial." The magnitude of $\vec{r}_{2\text{-}1}$ is the distance between the centers of the two objects.

We say that the gravitational constant G is "universal" because it is the same for any pair of interacting masses, no matter how big or small they are, or where they are located. Because G is universal, it can be measured for any pair of objects and then used with other pairs of objects. As we describe on page 97, Cavendish was the first person to make such a measurement.

$$-G\frac{m_2 m_1}{|\vec{r}_{2\text{-}1}|^2}\hat{r}_{2\text{-}1}$$

Figure 2.29 The gravitational force depends on the *product* of the two masses.

As highlighted in Figure 2.29, the gravitational force is proportional to the *product* of the two masses, $m_2 m_1$. If you double either of these masses, keeping the other one the same, the force will be twice as big. If you double both of the masses, the force will be four times as big. Since $m_2 m_1 = m_1 m_2$, the magnitude of the force exerted on object 1 by object 2 is exactly the same as the magnitude of the force exerted on object 2 by object 1 (but the direction is opposite).

$$-G\frac{m_2 m_1}{|\vec{r}_{2\text{-}1}|^2}\hat{r}_{2\text{-}1}$$

Figure 2.30 The gravitational force is an "inverse square" law.

The gravitational force is an "inverse square" law. As highlighted in Figure 2.30, the square of the center-to-center distance appears in the denominator. This means that the gravitational force depends very strongly on the distance between the objects. For example, if you double the distance between them, the only thing that changes is the denominator, which gets four times bigger (2 squared is 4), so the force is only 1/4 as big as before.

? If you move the masses 10 times farther apart than they were originally, how does the gravitational force change?

The force goes down by a factor of 100. Evidently when two objects are very far apart, the gravitational forces they exert on each other will be vanishingly small: big denominator, small force.

$$-G\frac{m_2 m_1}{|\vec{r}_{2\text{-}1}|^2}\hat{r}_{2\text{-}1}$$

Figure 2.31 The direction of the gravitational force on object 2 is in the opposite direction to the unit vector pointing *toward* object 2.

The minus sign with the unit vector highlighted in Figure 2.31 gives the direction of the force exerted on object 2 by object 1. The vector $\vec{r}_{2\text{-}1}$ points toward object 2, as does the unit vector $\hat{r}_{2\text{-}1}$, and the force acting on object 2 points in the opposite direction.

A useful way to think about the gravitational force law is to factor it into magnitude and direction, like this:

- A vector is a magnitude times a direction: $\vec{F}_{grav} = |\vec{F}_{grav}|\hat{F}_{grav}$
  - Magnitude: $|\vec{F}_{grav}| = G\dfrac{m_2 m_1}{|\vec{r}_{2\text{-}1}|^2}$
  - Direction (unit vector): $\hat{F}_{grav} = -\hat{r}_{2\text{-}1}$

It is usually simplest to calculate the magnitude and direction separately, then combine them to get the vector force. That way you can focus on one thing at a time rather than getting confused (or intimidated!) by the full complexity of the vector force law.

Ex. 2.18 Masses M and m attract each other with a gravitational force of magnitude F. Mass m is replaced with a mass $3m$, and it is moved four times farther away. Now what is the magnitude of the force?

Ex. 2.19 A 3 kg ball and a 5 kg ball are 2 m apart, center to center. What is the magnitude of the gravitational force that the 3 kg ball exerts on the 5 kg ball? What is the magnitude of the gravitational force that the 5 kg ball exerts on the 3 kg ball?

Ex. 2.20 Measurements show that the Earth's gravitational force on a mass of 1 kg near the Earth's surface is 9.8 N. The radius of the Earth is 6400 km (6.4×10^6 m). From these data determine the mass of the Earth.

2.7.2 Calculating the gravitational force on a planet

We'll go through a complete calculation to see in full gory detail how to evaluate the gravitational force law in the most general situation. In Figure 2.32 is a star of mass 4×10^{30} kg located at position $\langle 2\times10^{11}, 1\times10^{11}, 1.5\times10^{11}\rangle$ m and a planet of mass 3×10^{24} kg located at position $\langle 3\times10^{11}, 3.5\times10^{11}, -0.5\times10^{11}\rangle$ m. These are typical values for stars and planets. Notice that the mass of the star is much greater than that of the planet.

We will calculate the gravitational force exerted on the planet by the star. In Figure 2.32 we show the *x*, *y*, and *z* components of the positions, to be multiplied by 1×10^{11} m. Make sure you understand how the numbers on the diagram correspond to the positions given as vectors.

Here is a summary of the steps we will take to calculate the force acting on the planet due to the star:

- Calculate $\vec{r}_{2\text{-}1}$, the position of the planet relative to the star.
- Calculate $|\vec{r}_{2\text{-}1}|$, the (scalar) distance from star to planet.
- Calculate $Gm_2m_1/|\vec{r}_{2\text{-}1}|^2$, the magnitude of the force.
- Calculate $-\hat{r}_{2\text{-}1} = -\vec{r}_{2\text{-}1}/|\vec{r}_{2\text{-}1}|$, the direction of the force.
- Multiply the magnitude times the direction to get the vector force.

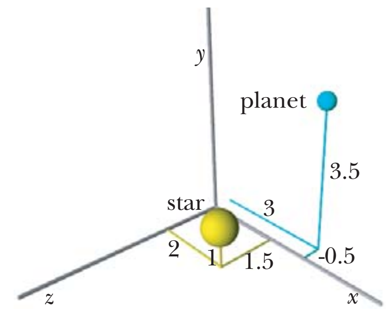


Figure 2.32 A star and a planet interact gravitationally. We will calculate the gravitational forces. The *x*, *y*, and *z* components are to be multiplied by 1×10^{11} m.

The relative position vector

An important quantity in the force law is the position of the center of the planet relative to the center of the star, $\vec{r}_{2\text{-}1}$, so we start by calculating this relative position vector.

? Think about what you know about calculating relative position vectors, and try to calculate $\vec{r}_{2\text{-}1}$ before reading ahead.

As usual, we just calculate "final minus initial":

$$\vec{r}_{2\text{-}1} = \vec{r}_2 - \vec{r}_1$$

$$\vec{r}_{2\text{-}1} = \langle 3\times10^{11}, 3.5\times10^{11}, -0.5\times10^{11}\rangle \text{ m} - \langle 2\times10^{11}, 1\times10^{11}, 1.5\times10^{11}\rangle \text{ m}$$

$$\vec{r}_{2\text{-}1} = \langle 1\times10^{11}, 2.5\times10^{11}, -2\times10^{11}\rangle \text{ m}$$

In Figure 2.33, the diagram shows that the signs of the components of $\vec{r}_{2\text{-}1}$ make sense (positive *x* and *y* components, negative *z* component), which is an important check that we haven't made any sign errors.

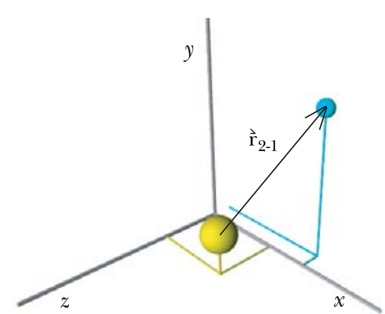


Figure 2.33 The position vector of the planet relative to the star.

The distance

In order to calculate the magnitude of the force, we'll need the distance $|\vec{r}_{2\text{-}1}|$ between the centers of the star and planet (and a bit later we'll also need $|\vec{r}_{2\text{-}1}|$ for calculating the unit vector).

? Try to calculate the magnitude of $\vec{r}_{2\text{-}1}$ before reading ahead.

As usual, we calculate the distance, which is a scalar, by using the 3D version of the Pythagorean theorem:

$$|\vec{r}_{2\text{-}1}| = \sqrt{(1\times10^{11})^2 + (2.5\times10^{11})^2 + (-2\times10^{11})^2} \text{ m}$$

$$|\vec{r}_{2\text{-}1}| = 3.35\times10^{11} \text{ m}$$

The magnitude of the force

? Before reading ahead, try to calculate the magnitude of the force, $\left|\vec{F}_{grav\text{ on planet by star}}\right|$.

The magnitude of the force on the planet by the star is this:

$$\left|\vec{F}_{grav\text{ on planet by star}}\right| = G\frac{m_2 m_1}{\left|\vec{r}_{2\text{-}1}\right|^2}$$

$$\left|\vec{F}_{grav\text{ on planet by star}}\right| = \left(6.7\times10^{-11}\ \frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\right)\frac{(3\times10^{24}\ \text{kg})(4\times10^{30}\ \text{kg})}{(3.35\times10^{11}\ \text{m})^2}$$

$$\left|\vec{F}_{grav\text{ on planet by star}}\right| = 7.16\times10^{21}\ \text{N}$$

This looks like a big force, but it's acting on a very big mass, so it isn't obvious whether this is really a "big" force in terms of what it will do.

The direction of the force

? Can you think how to calculate the vector direction of the gravitational force, the unit vector $\hat{F}_{grav\text{ on planet by star}}$? Try it.

We've got the magnitude of the force. We also need to calculate the direction of the force, which involves the unit vector:

$$\text{Direction of force: } \hat{F}_{grav\text{ on planet by star}} = -\hat{r}_{2\text{-}1} = -\frac{\vec{r}_{2\text{-}1}}{\left|\vec{r}_{2\text{-}1}\right|}$$

$$\hat{F}_{grav\text{ on planet by star}} = -\frac{\langle 1\times10^{11}, 2.5\times10^{11}, -2\times10^{11}\rangle\ \text{m}}{3.35\times10^{11}\ \text{m}}$$

$$\hat{F}_{grav\text{ on planet by star}} = \langle -0.299, -0.746, 0.597\rangle$$

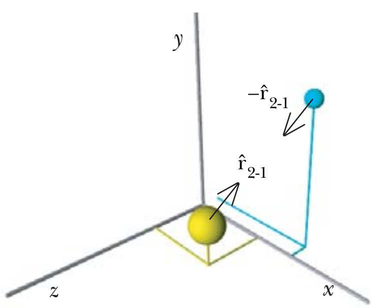


Figure 2.34 Unit vectors.

Notice that the unit vector giving the direction has no units, because the meters in the numerator and the denominator cancel. Figure 2.34 shows the unit vectors pointing from star toward planet, and from planet toward star.

The vector gravitational force

? Try to construct the vector force. You know the magnitude, and you know the direction.

Finally we can multiply the magnitude of the force times the unit vector direction of the force to get the full vector force:

$$\vec{F}_{grav\text{ on planet by star}} = \left|\vec{F}_{grav\text{ on planet by star}}\right|\hat{F}_{grav\text{ on planet by star}}$$

$$\vec{F}_{grav\text{ on planet by star}} = (7.16\times10^{21}\ \text{N})\langle -0.299, -0.746, 0.597\rangle$$

$$\vec{F}_{grav\text{ on planet by star}} = \langle -2.14\times10^{21}, -5.34\times10^{21}, 4.27\times10^{21}\rangle\ \text{N}$$

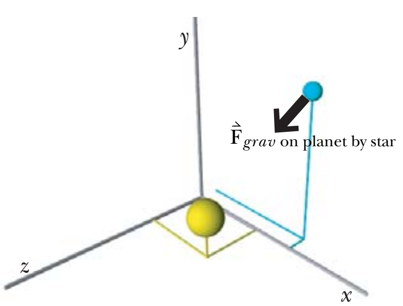


Figure 2.35 The gravitational force exerted on the planet by the star.

Checking our result

It is important to check long calculations such as this, because there are many opportunities to make a mistake along the way. There are several checks we can make.

- *Diagram:* An important check is to make a diagram like Figure 2.35 and see whether the direction of the calculated force makes sense. Pay particular attention to signs. The diagram shows that the force acting on the planet points in the $-x$, $-y$, $+z$ direction, and this agrees with our calculation, which has the same component signs.

- *Order of magnitude:* Make a rough "order of magnitude" check, dropping all the detailed numbers. In terms of "order of magnitude" the distance between star and planet is *very* roughly 1×10^{11} m, the mass of the star is *very* roughly 1×10^{30} kg, the mass of the planet is *very* roughly 1×10^{24} kg, and the gravitational constant is *very* roughly 1×10^{-11} $\mathrm{N\cdot m^2/kg^2}$. Therefore we expect the magnitude of the force to be *very* roughly this:

$$\left(1\times10^{-11}\ \frac{\mathrm{N\cdot m^2}}{\mathrm{kg^2}}\right)\frac{(1\times10^{24}\ \mathrm{kg})(1\times10^{30}\ \mathrm{kg})}{(1\times10^{11}\ \mathrm{m})^2} = 1\times10^{21}\ \mathrm{N}$$

  In this very rough order of magnitude calculation, ignoring numerical details, we just need to add and subtract exponents, and one hardly even needs a calculator. The fact that we get within an order of magnitude of the result 7.16×10^{21} N is evidence that we haven't made any huge mistakes.
- *Units:* An important check is that the units came out correctly, in newtons.
- *Unit vector:* Check to see whether the calculated unit vector does indeed have magnitude 1:

$$|\hat{\mathrm{F}}_{grav\ \mathrm{on\ planet\ by\ star}}| = \sqrt{(-0.299)^2+(-0.746)^2+(0.597)^2} = 1.001$$

  The magnitude isn't exactly 1 because we rounded off the intermediate calculations to three significant figures.

Our result passes all these checks. This doesn't prove we haven't made a mistake somewhere, but at least we've ruled out many possible errors.

The force on the star exerted by the planet

We've just calculated the force exerted on the planet by the star. To calculate the force exerted on the star by the planet, we could redo all these lengthy calculations in the same way, but that would be a lot of work.

? Look at the form of the gravitational force law and think carefully. Do you see a way to write down immediately the gravitational force on the star by the planet, without doing any additional calculations?

Note that the magnitude is exactly the same, because it involves exactly the same quantities. The only change is the direction of the force, which is in the opposite direction. Therefore we can immediately write the new result, just by flipping the signs:

$$\vec{\mathrm{F}}_{grav\ \mathrm{on\ star\ by\ planet}} = \langle 2.14\times10^{21}, 5.34\times10^{21}, -4.27\times10^{21}\rangle\ \mathrm{N}$$

Note that the signs of the force components are consistent with Figure 2.36. You might be puzzled that the planet pulls just as hard on the star as the star pulls on the planet. We'll discuss this in more detail later in this chapter.

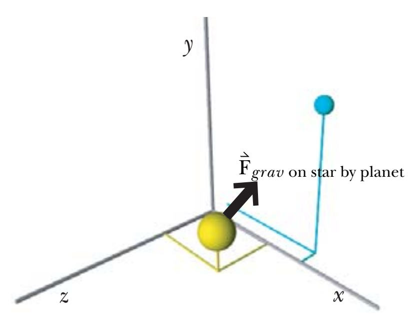


Figure 2.36 The gravitational force exerted on the star by the planet.

How can you remember all these steps?

Calculating the vector gravitational force requires many steps, but each step is pretty straight-forward. How can one remember what steps to take? Think of the equation for the gravitational force as a guide and outline for what to do.

$$\vec{\mathrm{F}}_{grav\ \mathrm{on\ 2\ by\ 1}} = -G\frac{m_2 m_1}{|\vec{\mathrm{r}}_{2\text{-}1}|^2}\hat{\mathrm{r}}_{2\text{-}1}$$

- When you look at the formula you see that you need $\vec{r}_{2\text{-}1}$, so you calculate it ("final minus initial").
- You also see that you need $|\vec{r}_{2\text{-}1}|$, so you calculate that.
- Now you see that you now have enough information to calculate the magnitude of the force.
- The formula tells you to calculate the negative of the unit vector, to get the direction.
- The formula says to multiply magnitude times direction. That's it!

With this scheme in mind, look over the previous few pages to review how we carried out the evaluation of the gravitational force law.

Ex. 2.21 The mass of the Earth is $6\times10^{24}\,\text{kg}$, and the mass of the Moon is $7\times10^{22}\,\text{kg}$. At a particular instant the Moon is at location $\langle 2.8\times10^{8}, 0, -2.8\times10^{8}\rangle$ m, in a coordinate system whose origin is at the center of the Earth.

(a) What is $\vec{r}_{\text{M-E}}$ the relative position vector from the Earth to the Moon?

(b) What is $|\vec{r}_{\text{M-E}}|$?

(c) What is the unit vector $\hat{r}_{\text{M-E}}$?

(d) What is the gravitational force exerted by the Earth on the Moon? Your answer should be a vector.

2.7.3 Using the gravitational force to predict motion

You now know that this star exerts this force on this planet at this instant, but what's the point in knowing this? By itself, this isn't particularly interesting. But if we use this calculated force in the Momentum Principle, we can predict the future motion of the planet.

- We know the force at this instant, and if we know the initial momentum $\vec{p}_i$ of the planet we can use the update form of the Momentum Principle, $\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t$, to predict the momentum of the planet a short time Δt later. We need to choose Δt short enough that the force doesn't change much during this brief time interval, due to changes in the positions of the interacting objects.
- We can also use the velocity to update the position of the planet, using the position update formula $\vec{r}_f = \vec{r}_i + \vec{v}_{\text{avg}}\Delta t$. Again, we need to choose Δt short enough that the velocity doesn't change much during that brief time interval, so that we can use the initial or final velocity as a good approximation to the average velocity.

After doing these momentum and position updates, we have successfully predicted the new momentum and position a short time into the future. Now we can repeat this process, calculating a new gravitational force based on the new position, updating the momentum, and updating the position. Step by step we can predict the motion of the planet into the future.

Although the massive star won't move very much, we could also update its momentum and position repetitively. As we've seen, it's easy to get the force on the star just by flipping signs.

Here is a summary of this iterative scheme:

- Start with initial positions and momenta of the interacting objects.
  - Calculate the (vector) gravitational forces acting on each object.
  - Update the momentum of each object: $\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t$.
  - Update the positions: $\vec{r}_f = \vec{r}_i + \vec{v}_{\text{avg}}\Delta t$, where $\vec{v}_{\text{avg}} \approx \vec{p}_f/m$.
  - Repeat.

In order to plan the trajectory for a spacecraft from Earth to a particular landing site on Mars, NASA has to carry out a very large number of such update calculations, taking small steps (small Δt) to achieve high accuracy. This includes calculating the net force as the vector sum of all the gravitational forces exerted on the spacecraft along the way by the Sun, Earth, Moon, Mars, and other objects in the Solar System. Doing all these force and update calculations by hand would be impossibly difficult, so people at NASA write computer programs to do these repetitive calculations. Later we will show you how to write such programs.

Updating the momentum and position of the planet

To see in detail how this iterative scheme works, we'll carry out one complete step—one update of the momentum and position. In order to start the iteration we have to know the initial positions and momenta of the star and planet. We already know their initial positions. Let's suppose we also know their initial velocities:

$$\vec{v}_{i,\text{star}} = \langle 0, 0, 0 \rangle \text{ m/s}$$

$$\vec{v}_{i,\text{planet}} = \langle 1\times10^4, 2\times10^4, 1.5\times10^4 \rangle \text{ m/s}$$

The star is initially at rest and, because it has a huge mass compared to the planet, it won't move very much. For simplicity we'll make the approximation that the star is fixed in position and never moves.

Choosing a sufficiently small time interval

We need to decide how big a value of Δt we can get away with and yet preserve adequate accuracy. The smaller the time step Δt, the more accurate the calculation, because the force and velocity won't change much during the brief time interval. However, taking smaller steps means doing more calculations to predict some time into the future.

Here is a way to choose a time interval. The initial speed is about $v \approx 3\times10^4$ m/s. Choose a Δt so that the distance the planet goes in this time interval, $v\Delta t$, is small compared to the distance d between star and planet, which is about 3×10^{11} m. As the planet changes position the gravitational force will change in magnitude and direction, as you can see in Figure 2.37. In the Momentum Principle $\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t$ we need for Δt to be small enough that $\vec{F}_{\text{net}}$ is nearly constant during this time interval.

Let's choose as a tentative criterion that the distance the planet moves ought to be about 0.001 times the distance from the star to the planet:

$$v\Delta t \approx 0.001\, d$$

$$\Delta t \approx \frac{0.001\, d}{v} = \frac{(0.001)(3\times10^{11}\text{ m})}{(3\times10^4\text{ m/s})} = 1\times10^4\text{ s}$$

This is about 3 hours. If instead we choose a large time interval of 1×10^7 s, say, in the first update the planet would move a long distance, about $(3\times10^4\text{ m/s})(1\times10^7\text{ s}) = 3\times10^{11}$ m, which is about as large as the distance to the star, and during that step the actual gravitational force would change a great deal in magnitude and direction. This would make our predictions very inaccurate.

Whenever you choose a time interval to use for updating momentum, make sure it is sufficiently small that the net force doesn't change very much in magnitude or direction during your chosen time interval.

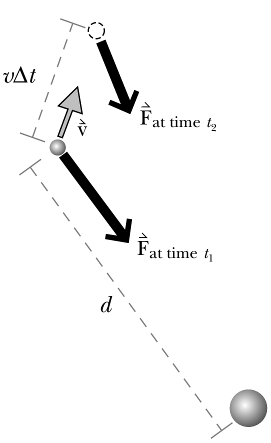


Figure 2.37 If Δt is too big, the gravitational force will change a lot in magnitude and direction during the time interval.

On the other hand, if the force happens to be constant in magnitude and direction, you can use as large a Δt as you like without making an error in the momentum update.

Update the momentum of the planet

? We know the force acting on the planet, and we've chosen a time interval to make one step in predicting the future of the planet. Try to predict the momentum the planet will have after 1×10^4 s.

The planet is moving at a high speed, about 3×10^4 m/s, but this is small compared to the speed of light, which is 3×10^8 m/s, so the initial momentum is $\vec{p}_i \approx m\vec{v}_i$.

$$\vec{p}_i \approx m\vec{v}_i = (3\times10^{24}\text{ kg})(\langle 1\times10^4, 2\times10^4, 1.5\times10^4\rangle\text{ m/s})$$

$$\vec{p}_i = \langle 3\times10^{28}, 6\times10^{28}, 4.5\times10^{28}\rangle\text{ kg}\cdot\text{m/s}$$

Use the Momentum Principle in its update form:

$$\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t$$

$$\vec{p}_f = \langle 3\times10^{28}, 6\times10^{28}, 4.5\times10^{28}\rangle\text{ kg}\cdot\text{m/s} + (\langle -2.14\times10^{21}, -5.34\times10^{21}, 4.27\times10^{21}\rangle\text{ N})(1\times10^4\text{ s})$$

$$\vec{p}_f = \langle 2.9979\times10^{28}, 5.9947\times10^{28}, 4.5043\times10^{28}\rangle\text{ kg}\cdot\text{m/s}$$

Notice that the momentum didn't change very much, because we deliberately chose a small time interval to ensure accuracy in the momentum update, by making sure that the force remained nearly constant during the brief time interval.

Update the position of the planet

? Think about how you would update the position of the planet.

We must use the position update formula, $\vec{r}_f = \vec{r}_i + \vec{v}_{\text{arg}}\Delta t$, but what is the average velocity $\vec{v}_{\text{arg}}$? The velocity (and momentum) changed during the 1×10^4 s time step. Should we use the initial velocity? The final velocity? The arithmetic average of these two velocities?

A key point is that the momentum changed very little, so the velocity changed very little. Therefore it hardly matters which velocity we use. To be concrete about this, and for simplicity in computer calculations, we'll normally use the final velocity (obtained from the just-calculated final momentum) to represent the average velocity. This works well as long as the time step is small. Since $\vec{p}_f \approx m\vec{v}_f$, $\vec{v}_f \approx \vec{p}_f/m$, and the position update is this:

$$\vec{r}_f \approx \vec{r}_i + \frac{\vec{p}_f}{m}\Delta t$$

$$\vec{r}_f = \left((\langle 3\times10^{11}, 3.5\times10^{11}, -0.5\times10^{11}\rangle\text{ m}) + \frac{(\langle 2.9979\times10^{28}, 5.9947\times10^{28}, 4.5043\times10^{28}\rangle\text{ kg}\cdot\text{m/s})}{(3\times10^{24}\text{ kg})}(1\times10^4\text{ s})\right)$$

$$\vec{r}_f = \langle 3.001\times10^{11}, 3.502\times10^{11}, -0.4985\times10^{11}\rangle\text{ m}$$

What have we done?

Whew! After all that computational drudgery, what have we accomplished? We've predicted the future of the planet!

Knowing the planet's initial position and momentum, we predict that after 1×10^4 s the planet will be in a new position that we've calculated, with a new momentum that we've calculated. So what? Well, we could do this again. The second time step would involve exactly the same calculations we just did, but the "initial" position and momentum would be the "final" ones we just calculated. We could keep going, doing this over and over and over, and predict as far ahead into the future as we are willing to do the tedious calculations.

You say you'd rather not do all those calculations? Right! The thing to do is to instruct a computer to do this for you. What you need to do is learn how to write out one set of instructions for the computer to do repetitively, over and over. Next we will show you how to do this, but it was important to guide you once through the gory details of the calculation, so that you could understand exactly what it is you have to tell the computer to do.

2.7.4 Telling a computer what to do

Here we summarize how to organize a computer program in order to tell a computer how to apply the Momentum Principle repetitively to predict the future.

- Define the values of constants such as G to use in the program.
- Specify the masses, initial positions, and initial momenta of the interacting objects.
- Specify an appropriate value for Δt, small enough that the objects don't move very far during one update.
- Create a "loop" structure for repetitive calculations:
  - Calculate the (vector) forces acting on each object.
  - Update the momentum of each object: $\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t$.
  - Update the positions: $\vec{r}_f = \vec{r}_i + \vec{v}_{\text{avg}}\Delta t$, where $\vec{v}_{\text{avg}} \approx \vec{p}_f / m$.
  - Repeat.

The details of the actual program statements depend on what programming language or system you use, and your instructor will provide you with the details of the particular tool you will use.

A computer calculation

In the preceding sections you took one step in such a calculation, by hand. Figure 2.38 shows the orbit of the planet as predicted by a computer program written using VPython (http://vpython.org) that continually updated the momentum and position of the planet, repeatedly using the gravitational force law, the Momentum Principle, and the position update formula. The star and planet are shown much larger than they really are, for visibility in the largely empty space.

Using the gravitational force law, Newton was able to explain the motion of planets and moons in the Solar System. Later scientists have found that it is possible to explain even the motions of other stars in our own and other galaxies with the same formula, so it appears that this is truly a universal law of gravitation, applying to all pairs of masses everywhere in the universe.

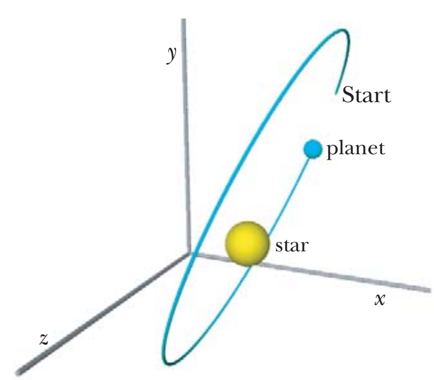


Figure 2.38 The motion of the planet around the star. The star and planet are shown much larger than they really are.

Vary the step size

After you get a program running, it is very important that you decrease the value of the step size Δt and make sure that the motion doesn't change. Of course the motion will take longer, because you're doing more calculations, but the issue is that the shape of the trajectory shouldn't change. If it does change significantly, that means that Δt was originally too big, leading to the net force and average velocity changing too much during the time Δt.

Keep decreasing Δt until you find that the shape of the trajectory hardly changes. Now you have an adequately small step size Δt.

Define and use constants

It is important to gather together at the start of the program the definitions of any constants such as G, and use these defined symbols everywhere else in the program (that is, use G rather than 6.7E–11 in later calculations). You should avoid entering actual numbers such as 6.7E–11 more than once in a program, since if there is a mistake in a constant you have to correct it in many places. Also, it is easier to read a program that uses G everywhere rather than the number 6.7E–11.

Why not just use calculus?

You might wonder why we don't simply use calculus to predict the motion of planets and stars. One answer is that we do use calculus. Step by step, we add up a large number of tiny increments of the momentum of a body to calculate a large change in its momentum over a long time, and this corresponds to an approximate evaluation of an integral (which is an infinite sum of infinitesimal amounts).

A more interesting answer is that the motion of most physical systems *cannot* be predicted using calculus in any way other than by this step-by-step approach. In a few cases calculus does give a general result without going through this procedure. For example, an object subjected to a constant force has a constant rate of change of momentum and velocity, and calculus can be used to obtain a prediction for the position as a function of time, as we saw in the example of a ball thrown through a vacuum. The elliptical orbits of two stars around each other can be predicted mathematically, although the math is quite challenging. But the general motion of three stars around each other has never been successfully analyzed in this way. The basic problem is that it is usually relatively easy to take the derivative of a known function, but it is often impossible to determine in algebraic form the integral of a known function, which is what would be involved in long-term prediction.

It isn't a question of taking more math courses in order to be able to solve the "three-body" problem: there is no general mathematical solution. However, a step-by-step procedure of the kind we carried out for the planet can easily be extended to three or more bodies. Just calculate all the forces acting between pairs of bodies, update the momenta, and update the positions. This is why we study the step-by-step prediction method in detail, because it is a powerful technique of increasing importance in modern science and engineering, thanks to the availability of powerful computers to do the repetitive work for us.

Ex. 2.22 The Earth goes around the Sun in 365 days, in a nearly circular orbit. In a computer calculation of the orbit (which is actually an ellipse), approximately how big can Δt be and still get good accuracy in the prediction of the motion?

Ex. 2.23 A pendulum swings with a "period" (time for one round trip) of 2 s. In a computer calculation of the motion, approximately how big can Δt be and still get good accuracy in the prediction of the motion?

2.7.5 Approximate gravitational force near the Earth's surface

Earlier we used the expression mg to represent the magnitude of the gravitational force on an object near the Earth's surface. This is an approximation to the actual force, but it is a good one. The magnitude of the gravitational force that the Earth exerts on an object of mass m near the Earth's surface (Figure 2.39) is

$$F_g = G\frac{M_E m}{(R_E + y)^2}$$

where y is the distance of the object above the surface of the Earth, R_E is the radius of the Earth, and M_E is the mass of the Earth.

A spherical object of uniform density can be treated as if all its mass were concentrated at its center. As a result, we can treat the effect of the Earth on the object as though the Earth were a tiny, very dense ball a distance $R_E + y$ away.

The gravitational force exerted by the Earth on an atom at the top of the object is slightly different from the force on an atom at the bottom of the object, because each atom is a slightly different distance from the center of the Earth. How much can this difference really matter in an analysis? Suppose the height of the object is a meter, and the bottom of the object is one meter above the surface. The radius of the Earth is about 6.4×10^6 m. Then

$$F_{\text{top}} = G\frac{M_E m}{(6400002\ \text{m})^2} \text{ whereas } F_{\text{bottom}} = G\frac{M_E m}{(6400001\ \text{m})^2}$$

For most purposes this difference is not significant. In fact, for all interactions of objects near the surface of the Earth, it makes sense to use the same approximate value, R_E, for the distance from the object to the center of the Earth. This simplifies calculation of gravitational forces by allowing us to combine all the constants into a single lumped constant, g:

$$F_g \approx \left(G\frac{M_E}{R_E^2}\right)m = gm\text{, so that } g = G\frac{M_E}{R_E^2}$$

The constant g, called "the magnitude of the gravitational field," has the value $g = +9.8$ newtons/kilogram near the Earth's surface. The "gravitational field" $\vec{g}$ at a location in space is defined to be the (vector) gravitational force that would be exerted on a 1 kg mass placed at that location. A 2 kg mass would experience a force twice as large. In general, a mass m will experience a force $m\vec{g}$ of magnitude mg.

Note that g is a *positive* number, the magnitude of the gravitational field. We can use this approximate formula for gravitational force in our analysis of any interactions occurring near the surface of the Earth.

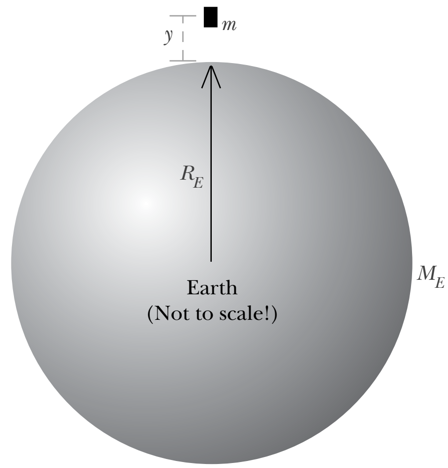


Figure 2.39 Determining the gravitational force of the Earth on an object a height y above the surface.

Ex. 2.24 Show that GM_E/R_E^2 equals 9.8 N/kg. The mass of the Earth is 6×10^{24} kg.

Ex. 2.25 At what height above the surface of the Earth is there a 1% difference between the approximate magnitude of the gravitational field (9.8 N/kg) and the actual magnitude of the gravitational field at that location? That is, at what height y above the Earth's surface is $GM_E/(R_E + y)^2 = 0.99\,GM_E/R_E^2$?

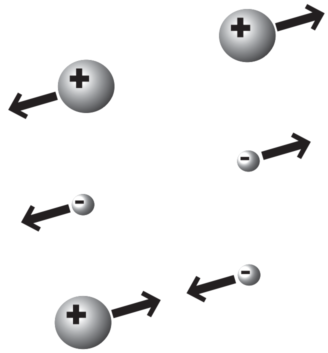


Figure 2.40 Protons repel each other; electrons repel each other; protons and electrons attract each other.

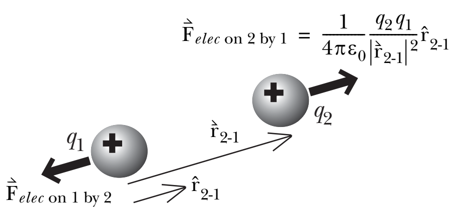


Figure 2.41 The electric force exerted on object 2 by object 1. (The force exerted on object 2 by object 1 has the same magnitude but opposite direction.)

2.8 The electric force law: Coulomb's law

There are electric interactions between "charged" particles such as the protons and electrons found in atoms. It is observed that two protons repel each other, as do two electrons, while a proton and an electron attract each other (Figure 2.40; where protons are said to have "positive electric charge" and electrons have "negative electric charge").

The force corresponding to this electric interaction is similar to the gravitational force law, and it is known as "Coulomb's law" to honor the French scientist who established this law in the late 1700's, during the same period when Cavendish measured the gravitational constant G. See Figure 2.41.

THE ELECTRIC FORCE LAW (COULOMB'S LAW)

$$\vec{F}_{elec\ on\ 2\ by\ 1} = \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_1}{|\vec{r}_{2\text{-}1}|^2}\hat{r}_{2\text{-}1}$$

where $\frac{1}{4\pi\varepsilon_0}$ is a universal constant: $\frac{1}{4\pi\varepsilon_0} = 9\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}$

Both the gravitational and electric forces are proportional to the inverse square of the center-to-center distance ("inverse square" laws). The universal electric constant (read as "one over four pi epsilon-zero") is very much larger than the gravitational constant, reflecting the fact that electric interactions are intrinsically much stronger than gravitational interactions. For example, consider a heavy weight that hangs from a thin metal wire. The small number of atoms interacting electrically in the wire have as big an effect on the hanging weight as a very much larger number of atoms in the huge Earth, interacting gravitationally.

The charges q_1 and q_2 must be measured in SI units called "coulombs." The proton has a charge of $+1.6\times10^{-19}$ C and the electron has a charge of -1.6×10^{-19}.

? Study the three cases in Figure 2.40. Why is no minus sign needed in the force law, unlike the case with the gravitational force law?

The difference is that two positively charged particles such as two protons repel each other, whereas masses are always positive but the gravitational force is attractive. Two negatively charged particles such as electrons also attract each other and minus times minus gives plus. Only if the two particles have opposite charges will they attract each other, and then the factor $q_2 q_1$ contributes the necessary minus sign, just as in the gravitational force law.

2.8.1 Interatomic forces

When two objects touch each other they exert forces on each other. At the microscopic level, these contact forces are due to electric interactions between the protons and electrons in one object and the protons and electrons in the other object. In Chapter 3 we will examine interatomic forces in more detail.

Ex. 2.26 A proton and an electron are separated by 1×10^{-10} m, the radius of a typical atom. Calculate the magnitude of the electric force that the proton exerts on the electron, and the magnitude of the electric force that the electron exerts on the proton.

2.9 Reciprocity

An important aspect of the gravitational and electric interactions (including the electric forces of atoms in contact with each other) is that the force that object 1 exerts on object 2 is equal and opposite to the force that object 2 exerts on object 1 (Figure 2.42). That the magnitudes must be equal is clear from the algebraic form of the laws, because $m_1 m_2 = m_2 m_1$ and $q_1 q_2 = q_2 q_1$. The directions of the forces are along the line connecting the centers, and in opposite directions.

This property is called "reciprocity" or "Newton's third law of motion":

RECIPROCITY

$$\vec{F}_{\text{on 1 by 2}} = -\vec{F}_{\text{on 2 by 1}} \text{ (gravitational and electric forces)}$$

The force that the Earth exerts on the massive Sun is just as big as the force that the Sun exerts on the Earth, so in the same time interval the momentum changes are equal in magnitude and opposite in direction ("equal and opposite" for short):

$$\Delta\vec{p}_1 = \vec{F}_{\text{on 1 by 2}}\Delta t = -\Delta\vec{p}_2$$

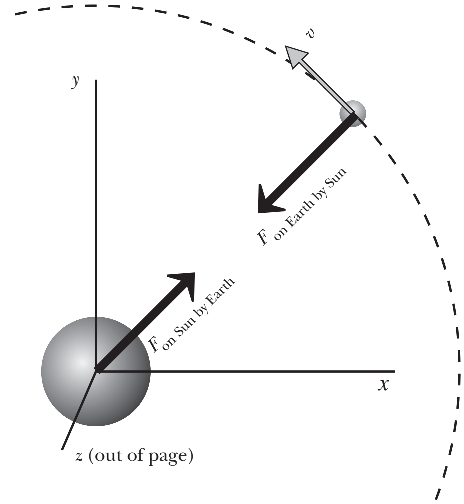


Figure 2.42 The Sun and the Earth exert equal and opposite forces on each other.

However, the velocity change $\Delta\vec{v} = \Delta\vec{p}/m$ of the Sun is extremely small compared to the velocity change of the Earth, because the mass of the Sun is enormous in comparison with the mass of the Earth. The mass of our Sun, which is a rather ordinary star, is 2×10^{30} kg. This is enormous compared to the mass of the Earth (6×10^{24} kg) or even to the mass of the largest planet in our Solar System, Jupiter (2×10^{27} kg). Nevertheless, very accurate measurements of small velocity changes of distant stars have been used to infer the presence of unseen planets orbiting those stars.

Magnetic forces do not have the property of reciprocity. Two electrically charged particles that are both moving can interact magnetically as well as electrically, and the magnetic forces that these two particles exert on each other need not be equal in magnitude nor opposite in direction. Reciprocity applies to gravitational and electric forces, but not in general to magnetic forces (except in some special cases).

Why reciprocity?

The algebraic forms of the gravitational and electric force laws indicate that reciprocity should hold. A diagram may be helpful in explaining *why* the forces behave this way.

Consider two small objects, a 3 gram object made up of three 1 gram balls, and a 2 gram object made up of two 1 gram balls (Figure 2.43). The distance between centers of the two objects is large compared to the size of either object, so the distances between pairs of 1 gram balls are about the same for all pairs.

You can see that in the 3 gram object each ball has two gravitational forces exerted on it by the distant balls in the 2 gram object.

? How many forces act on the 3 gram object?

There is a net force of $3 \times 2 = 6$ times the force associated with one pair of balls.

? Similarly, consider the forces acting on the 2 gram object. How many forces act on the 2 gram object?

There is again a net force of $2 \times 3 = 6$ times the force associated with one pair of balls.

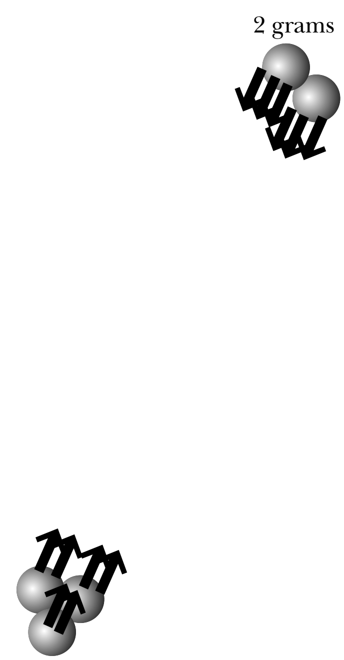


Figure 2.43 Reciprocity: there are 6 forces acting on the 3 gram object and 6 forces acting on the 2 gram object.

The effect is that the force exerted by the 3 gram object on the 2 gram object has the same magnitude as the force exerted by the 2 gram object on the 3 gram object. The same reciprocity holds, for the same reasons, for the electric forces between a lithium nucleus containing 3 protons and a helium nucleus containing 2 protons.

2.10 The Newtonian synthesis

Newton devised a particular explanatory scheme in which the analysis of motion is divided into two distinct parts:

1) Quantify the interaction in terms of a concept called "force." Specific examples are Newton's law of gravitation and Coulomb's law.
2) Quantify the change of motion in terms of the change in a quantity called "momentum." The change in the momentum is equal to the force times Δt.

This scheme, called the "Newtonian synthesis," has turned out to be extraordinarily successful in explaining a huge variety of diverse physical phenomena, from the fall of an apple to the orbiting of the Moon. Yet we have no way of asking whether the Universe "really" works this way. It seems unlikely that the Universe actually uses the human concepts of "force" and "momentum" in the unfolding motion of an apple or the Moon.

We refer to the "Newtonian synthesis" both to identify and honor the particular, highly successful analysis scheme introduced by Newton, but also to remind ourselves that this is not the only possible way to view and analyze the universe.

Einstein's alternative view

Newton stated his gravitational force law but could give no explanation for it. He was content with showing that it correctly predicted the motion of the planets, and this was a huge advance, the real beginning of modern science.

Einstein made another huge advance by giving a deeper explanation for gravity, as a part of his general theory of relativity. He realized that the massive Sun bends space and time (!) in such a way as to make the planets move the way they do. The equations in Einstein's general theory of relativity make it possible to calculate the curvature of space and time due to massive objects, and to predict how other objects will move in this altered space and time.

Moreover, Einstein's theory of general relativity accurately predicts some tiny effects that Newton's gravitational law does not, such as the slight bending of light as it passes near the Sun. General relativity also explains some bizarre large-scale phenomena such as black holes and the observed expansion of the space between the galaxies.

Einstein's earlier special theory of relativity established that nothing, not even information, can travel faster than light. Because Newton's gravitational force formula depends only on the distance between objects, not on the time, something's wrong with the formula, since this implies that if an object were suddenly yanked away, its force on another object would vanish instantaneously, thus giving (in principle) a way to send information from one place to another instantaneously. Einstein's theory of general relativity doesn't have this problem.

Since the equations of general relativity are very difficult to work with, and Newton's gravitational law works very well for most purposes, in this course we will use Newton's approach to gravity. But you should be aware that for the most precise calculations one must use the theory of general relativity. For example, the highly accurate atomic clocks in the satellites that make up the Global Positioning System (GPS) have to be continually corrected using Einstein's theory of general relativity. Otherwise GPS positions would be wrong by several kilometers after just one day of operation!

2.11 *Derivation of special average velocity result

Here we offer two proofs, one geometric and one algebraic (using calculus), for the following special-case result concerning average velocity:

AVERAGE VELOCITY (SPECIAL CASE)

$$v_{\text{avg},x} = \frac{(v_{xi} + v_{xf})}{2} \text{ } \textit{only} \text{ if } v_x \text{ changes at a constant rate}$$

$(F_{\text{net},x}$ is constant, $v << c)$; similar results for $v_{\text{avg,y}}$ and $v_{\text{avg,z}}$

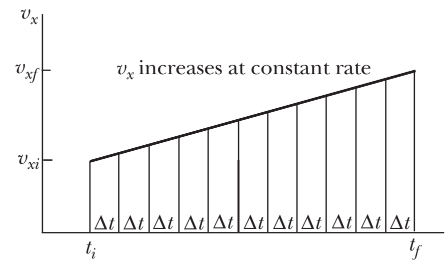


Figure 2.44 In this graph of v_x vs. t (constant force), form narrow vertical slices each of height v_x and width Δt.

If $F_{\text{net},x}$ is constant, $p_{xf} = p_{xi} + F_{\text{net, x}}\Delta t$ implies that p_x changes at a constant rate. At speeds small compared to the speed of light, $v_x \approx p_x/m$, so a graph of v_x vs. time is a straight line, as in Figure 2.44. In Figure 2.44 we form narrow vertical slices, each of height v_x and narrow width Δt.

Within each narrow slice v_x changes very little, so the change in position during the brief time Δt is approximately $\Delta x = v_x\Delta t$. Therefore the change in x is approximately equal to the area of the slice of height v_x and width Δt (Figure 2.45).

If we add up the areas of all these slices, we get approximately the area under the line in Figure 2.44, and this is also equal to the total displacement $\Delta x_1 + \Delta x_2 + \Delta x_3 + ... = x_f - x_i$. If we go to the limit of an infinite number of slices, each with infinitesimal width, the sum of slices really *is* the area, and this area we have shown to be equal to the change in position. This kind of sum of an infinite number of infinitesimal pieces is called an "integral" in calculus.

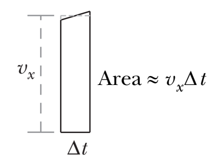


Figure 2.45 One narrow slice has an area given approximately by $v_x\Delta t$. This is equal to Δx, the displacement of the object.

The area under the line is a trapezoid, and from geometry we know that the area of a trapezoid is the average of the two bases times the altitude. Turn Figure 2.44 on its side, as in Figure 2.46, and you see that the top and bottom have lengths v_{xi} and v_{xf}, while the altitude of the trapezoid is the total time $(t_f - t_i)$. Therefore we have the following result:

$$\text{Trapezoid area} = x_f - x_i = \frac{(v_{xi} + v_{xf})}{2}(t_f - t_i)$$

Dividing by $(t_f - t_i)$, we have this:

$$\frac{x_f - x_i}{t_f - t_i} = \frac{(v_{xi} + v_{xf})}{2}$$

But by definition the x component of average velocity is the change in x divided by the total time, so we have proved that

$$v_{\text{avg},x} = \frac{(v_{xi} + v_{xf})}{2} \text{ } \textit{only} \text{ if } v_x \text{ changes linearly with time } (F_{\text{net},x} \text{ is constant})$$

$$\text{Area} = \frac{(\text{Top} + \text{Bottom})}{2}\text{Altitude}$$

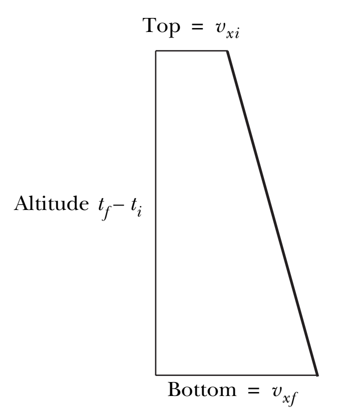


Figure 2.46 The area of the whole trapezoid is equal to the total displacement. The figure in Figure 2.44 has been turned on its side.

The proof depended critically on the straight-line ("linear") change in velocity, which occurs only if $F_{\text{net},x}$ is constant (and $v << c$). Otherwise we wouldn't have a trapezoidal area. That's why the result isn't true in general; it's only true in this important but special case.

Algebraic proof using calculus

An algebraic proof using calculus can also be given. We will use the x component of the derivative version of the Momentum Principle (more about this in the next chapter):

$$\Delta\vec{p} = \vec{F}_{\text{net}}\Delta t \text{ implies that } \frac{\Delta\vec{p}}{\Delta t} = \vec{F}_{\text{net}}$$

In the limit we have $\lim_{\Delta t \to 0} \frac{\Delta \vec{p}}{\Delta t} = \frac{d\vec{p}}{dt} = \vec{F}_{net}$ and $\frac{dp_x}{dt} = F_{net,x}$

If $F_{net,x}$ is a constant, the time derivative of p_x is a constant, so we have

$$p_x = F_{net,x}t + p_{xi} \text{ since } p_x = p_{xi} \text{ when } t = 0$$

You can check this by taking the derivative with respect to time t, which gives the original equation $dp_x/dt = F_{net,x}$. At speeds small compared to the speed of light, $v_x \approx p_x/m$, so we can write

$$v_x = \frac{F_{net,x}}{m}t + v_{xi} \text{ since } v_x = v_{xi} \text{ when } t = 0$$

But the x component of velocity is the rate at which x is changing:

$$v_x = \frac{dx}{dt} = \frac{F_{net,x}}{m}t + v_{xi}$$

Now the question is, can you think of a function of x that has this time derivative? Since the time derivative of t^2 is $2t$, the following formula for x has the appropriate derivative:

$$x = \frac{1}{2}\frac{F_{net,x}}{m}t^2 + v_{xi}t + x_i \text{ since } x = x_i \text{ when } t = 0$$

You can check this by taking the derivative with respect to t, which gives the equation for v_x, since $d(\frac{1}{2}t^2)/dt = t$ and $d(t)/dt = 1$.

The average velocity which we seek is the change in position divided by the total time:

$$v_{avg,x} = \frac{x - x_i}{t} = \frac{1}{2}\frac{F_{net,x}}{m}t + v_{xi} = \frac{1}{2}(v_{xf} - v_{xi}) + v_{xi}$$

where we have used the equation we previously derived for the velocity:

$$v_{xf} = v_x = \frac{F_{net,x}}{m}t + v_{xi}$$

Simplifying the expression for $v_{avg,x}$ we have the proof:

$$v_{avg,x} = \frac{(v_{xi} + v_{xf})}{2} \text{ \textit{only} if } v_x \text{ changes at a constant rate } (F_{net,x} \text{ is constant})$$

2.12 *Points and spheres

The gravitational force law applies to objects that are "point-like" (very small compared to the center-to-center distance between the objects). In the second volume of this textbook we will be able to show that any hollow spherical shell with uniform density acts gravitationally on external objects as though all the mass of the shell were concentrated at its center. The density of the Earth is not uniform, because the central iron core has higher density than the outer layers. But by considering the Earth as layers of hollow spherical shells, like an onion, with each shell of nearly uniform density, we get the result that the Earth can be modeled for most purposes as a point mass located at the center of the Earth (for very accurate calculations one must take into account small irregularities in the Earth's density from place to place). Similar statements can be made about other planets and stars. In Figure 2.47, the gravitational force is correctly calculated using the center-to-center distance $|\vec{r}_{2\text{-}1}|$.

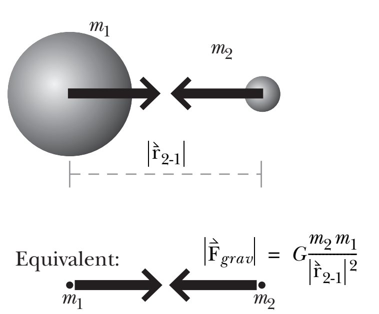


Figure 2.47 A sphere whose density depends only on distance from the center acts like a point particle with all the mass at the center.

This is not an obvious result. After all, in Figure 2.47 some of the atoms are closer and some farther apart than the center-to-center distance $|\vec{r}_{2\text{-}1}|$, but the net effect after adding up all the interactions of the individual atoms is as though the two objects had collapsed down to points at their centers. This is a very special property of $1/r^2$ forces, both gravitational and electric, and is not true for forces that have a different dependence on distance.

2.13 *Measuring the universal gravitational constant *G*

In order to make quantitative predictions and analyses of physical phenomena involving gravitational interactions, it is necessary to know the universal gravitational constant *G*. In 1797-1798 Henry Cavendish performed the first experiment to determine a precise value for *G* (Figure 2.48). In this kind of experiment, a bar with metal spheres at each end is suspended from a thin quartz fiber which constitutes a "torsional" spring. From other measurements, it is known how large a tangential force measured in newtons is required to twist the fiber through a given angle. Large balls are brought near the suspended spheres, and one measures how much the fiber twists due to the gravitational interactions between the large balls and the small spheres.

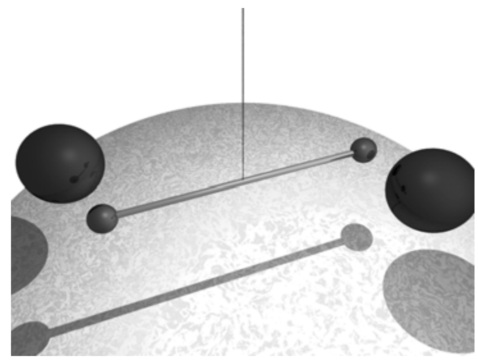


Figure 2.48 The Cavendish experiment.

If the masses are measured in kilograms, the distance in meters, and the force in newtons, the gravitational constant *G* has been measured in such experiments to be

$$G = 6.7\times10^{-11}\ \frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}$$

This extremely small number reflects the fact that gravitational interactions are inherently very weak compared with electromagnetic interactions. The only reason that gravitational interactions are significant in our daily lives is that objects interact with the entire Earth, which has a huge mass. It takes sensitive measurements such as the Cavendish experiment to observe gravitational interactions between two ordinary-sized objects.

2.14 *The Momentum Principle is valid only in inertial frames

Newton's first law is valid only in an "inertial frame" of reference, one in uniform motion (or at rest) with respect to the pervasive "cosmic microwave background" (see optional discussion at the end of Chapter 1). Since the Momentum Principle is a quantitative version of Newton's first law, we expect the Momentum Principle to be valid in an inertial reference frame, but not in a reference frame that is not in uniform motion. Let's check that this is true.

If you view some objects from a space ship that is moving uniformly with velocity $\vec{v}_s$ with respect to the cosmic microwave background, all of the velocities of those objects have the constant $\vec{v}_s$ subtracted from them, as far as you are concerned. For example, a rock moving at the same velocity as your spacecraft would have $\vec{v}_{rock} = (\vec{v}-\vec{v}_s) = \vec{0}$ in your reference frame: it would appear to be stationary as it coasted along beside your spacecraft.

With a constant spaceship velocity, we have $\Delta\vec{v}_s = \vec{0}$, and the change of momentum of the moving object reduces to the following (for speeds small compared to c):

$$\Delta[m(\vec{v}-\vec{v}_s)] = \Delta(m\vec{v}) - \Delta(m\vec{v}_s) = \Delta(m\vec{v}) \text{ since } \Delta(m\vec{v}_s) = \vec{0}$$

$$\text{Therefore, } \Delta[m(\vec{v}-\vec{v}_s)] = \Delta(m\vec{v}) = \vec{F}_{net}\Delta t$$

If the velocity $\vec{v}_s$ of the space ship doesn't change (it represents an inertial frame of reference), the form (and validity) of the Momentum Principle is unaffected by the motion of the space ship.

However, if your space ship increases its speed, or changes direction, $\Delta\vec{v}_s \neq \vec{0}$, an object's motion relative to you changes without any force acting on it. In that case the Momentum Principle is not valid for the object, because you are not in an inertial frame. Although the Earth is not an inertial frame because it rotates, and goes around the Sun, it is close enough to being an inertial frame for many everyday purposes.

2.15 *Updating position at high speed

If $v << c$, $\vec{p} \approx m\vec{v}$ and $\vec{v} \approx \vec{p}/m$. But at high speed it is more complicated to determine the velocity from the (relativistic) momentum. Here is a way to solve for $\vec{v}$ in terms of $\vec{p}$:

$$|\vec{p}| = \frac{1}{\sqrt{1-(|\vec{v}|/c)^2}}m|\vec{v}|$$

Divide by m and square: $\dfrac{|\vec{p}|^2}{m^2} = \dfrac{|\vec{v}|^2}{1-(|\vec{v}|/c)^2}$

Multiply by $(1-(|\vec{v}|/c)^2)$: $\dfrac{|\vec{p}|^2}{m^2} - \left(\dfrac{|\vec{p}|^2}{m^2c^2}\right)|\vec{v}|^2 = |\vec{v}|^2$

Collect terms: $\left(1+\dfrac{|\vec{p}|^2}{m^2c^2}\right)|\vec{v}|^2 = \dfrac{|\vec{p}|^2}{m^2}$

$$|\vec{v}| = \frac{|\vec{p}|/m}{\sqrt{1+\left(\frac{|\vec{p}|}{mc}\right)^2}}$$

But since $\vec{p}$ and $\vec{v}$ are in the same direction, we can write this:

$$\vec{v} = \frac{\vec{p}/m}{\sqrt{1+\left(\frac{|\vec{p}|}{mc}\right)^2}}$$

THE RELATIVISTIC POSITION UPDATE FORMULA

$$\vec{r}_f = \vec{r}_i + \frac{1}{\sqrt{1+\left(\frac{|\vec{p}|}{mc}\right)^2}}\left(\frac{\vec{p}}{m}\right)\Delta t \quad \text{(for small } \Delta t\text{)}$$

Note that at low speeds $|\vec{p}| \approx m|\vec{v}|$, and the denominator is $\sqrt{1+\left(\frac{|\vec{v}|}{c}\right)^2} \approx 1$, so the formula becomes the familiar $\vec{r}_f = \vec{r}_i + (\vec{p}/m)\Delta t$.

2.16 *Definitions, measurements, and units

Using the Momentum Principle requires a consistent way to measure length, time, mass, and force, and a consistent set of units. We state the definitions of the standard Système Internationale (SI) units, and we briefly discuss some subtle issues underlying this choice of units.

Units: meters, seconds, kilograms, and newtons

Originally the meter was defined as the distance between two scratches on a platinum bar in a vault in Paris, and a second was 1/86,400th of a "mean solar day." Now however the second is defined in terms of the frequency of light emitted by a cesium atom, and the meter is defined as the distance light travels in 1/299,792,458th of a second, or about 3.3×10^{-9} seconds (3.3 nanoseconds). The speed of light is defined to be exactly 299,792,458 m/s (very close to 3×10^{8} m/s). As a result of these modern redefinitions, it is really speed (of light) and time that are the internationally agreed-upon basic units, not length and time.

By international agreement, one kilogram is the mass of a platinum block kept in that same vault in Paris. As a practical matter, other masses are compared to this standard kilogram by using a balance-beam or spring weighing scale (more about this in a moment). The newton, the unit of force, is defined as that force which acting for 1 second imparts to 1 kilogram a velocity change of 1 m/s. We could make a scale for force by calibrating the amount of stretch of a spring in terms of newtons.

Some subtle issues

What we have just said about SI units is sufficient for practical purposes to predict the motion of objects, but here are some questions that might bother you. Is it legitimate to measure the mass that appears in the Momentum Principle by seeing how that mass is affected by gravity on a balance-beam scale? Is it legitimate to use the Momentum Principle to define the units of force, when the concept of force is itself associated with the same law? Is this all circular reasoning, and the Momentum Principle merely a definition with little content? Here is a chain of reasoning that addresses these issues.

Measuring inertial mass

When we use balance-beam or spring weighing scales to measure mass, what we're really measuring is the "gravitational mass," that is, the mass that appears in the law of gravitation and is a measure of how much this object is affected by the gravity of the Earth. In principle, it could be that this "gravitational mass" would be different from the so-called "inertial mass"—the mass that appears in the definition of momentum. It is possible to compare the inertial masses of two objects, and we find experimentally that inertial and gravitational mass seem to be entirely equivalent.

Here is a way to compare two inertial masses directly, without involving gravity. Starting from rest, pull on the first object with a spring stretched by some amount s for an amount of time Δt, and measure the increase of speed Δv_1. Then, starting from rest, pull on the second object with the same spring stretched by the same amount s for the same amount of time Δt, and measure the increase of speed Δv_2. We *define* the ratio of the inertial masses as $m_1/m_2 = \Delta v_2/\Delta v_1$. Since one of these masses could be the standard kilogram kept in Paris, we now have a way of measuring inertial mass in kilograms. Having defined inertial mass this way, we find experimentally that the Momentum Principle is obeyed by both of these objects in all situations, not just in the one special experiment we used to compare the two masses.

Moreover, we find to extremely high precision that the inertial mass in kilograms measured by this comparison experiment is exactly the same as the gravitational mass in kilograms obtained by comparing with a standard kilogram on a balance-beam scale (or using a calibrated spring scale), and that it doesn't matter what the objects are made of (wood, copper, glass, etc.). This justifies the convenient use of ordinary weighing scales to determine inertial mass.

Is this circular reasoning?

The definitions of force and mass may sound like circular reasoning, and the Momentum Principle may sound like just a kind of definition, with no real content, but there is real power in the Momentum Principle. Forget for a moment the definition of force in newtons and mass in kilograms. The experimental fact remains that any object if subjected to a single force by a spring with constant stretch experiences a change of momentum (and velocity) proportional to the duration of the interaction. Note that it is not a change of *position* proportional to the time (that would be a constant speed), but a change of *velocity*. That's real content. Moreover, we find that the change of velocity is proportional to the amount of stretch of the spring. That too is real content.

Then we find that a different object undergoes a different rate of change of velocity with the same spring stretch, but after we've made one single comparison experiment to determine the mass relative to the standard kilogram, the Momentum Principle works in all situations. That's real content.

Finally come the details of setting standards for measuring force in newtons and mass in kilograms, and we use the Momentum Principle in helping set these standards. But logically this comes after having established the law itself.

2.17 Summary

Fundamental Physical Principles

THE MOMENTUM PRINCIPLE

$$\Delta\vec{p} = \vec{F}_{net}\Delta t \text{ (for a short enough time interval } \Delta t)$$

$$\text{Update form: } \vec{p}_f = \vec{p}_i + \vec{F}_{net}\Delta t$$

THE SUPERPOSITION PRINCIPLE

The net force on an object is the vector sum of the individual forces exerted on it by all other objects. Each individual interaction is unaffected by the presence of other interacting objects.

Major new concepts

To apply the Momentum Principle to analyze the motion of a real-world system, several steps are required:

1. Choose a system, consisting of a portion of the Universe. The rest of the Universe is called the surroundings.
2. List objects in the surroundings that exert significant forces on the chosen system, and make a labeled diagram showing the external forces exerted by the objects in the surroundings.
3. Apply the Momentum Principle to the chosen system:

   $$\vec{p}_f = \vec{p}_i + \vec{F}_{net}\Delta t$$

   For each term in the Momentum Principle, substitute any values you know.
4. Apply the position update formula, if necessary:

   $$\vec{r}_f = \vec{r}_i + \vec{v}_{avg}\Delta t$$

5. Solve for any remaining unknown quantities of interest.
6. Check for reasonableness (units, etc.).

A labeled diagram (step 3) gives a physics view of the situation, and it defines symbols to use in writing an algebraic statement of the Momentum Principle.

System is a portion of the Universe acted on by the *surroundings*.

Force is a quantitative measure of interactions; units are newtons.

Impulse is the product of force times time $\vec{F}\Delta t$; momentum change equals net impulse (the impulse due to the net force).

Four fundamental types of interaction have been identified:
- gravitational interactions (all objects attract each other gravitationally)
- electromagnetic interactions (electric and magnetic interactions, closely related to each other); interatomic forces are electric in nature
- "strong" interactions (inside the nucleus of an atom)
- "weak" interactions (neutron decay, for example)

Physical models are tractable approximations/idealizations of the real world.

Computer prediction of motion:

- Define the values of constants such as G to use in the program.
- Specify the masses, initial positions, and initial momenta of the interacting objects.
- Specify an appropriate value for Δt, small enough that the objects don't move very far during one update.
- Create a "loop" structure for repetitive calculations:
  - Calculate the (vector) forces acting on each object.
  - Update the momentum of each object: $\vec{p}_f = \vec{p}_i + \vec{F}_{net}\Delta t$.
  - Update the positions: $\vec{r}_f = \vec{r}_i + \vec{v}_{arg}\Delta t$, where $\vec{v}_{arg} \approx \vec{p}_f / m$.
  - Repeat.

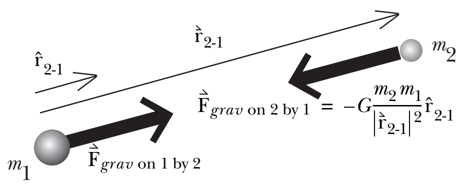


Figure 2.49 The gravitational force exerted on object 2 by object 1. (The force exerted on object 1 by object 2 has the same magnitude but opposite direction.)

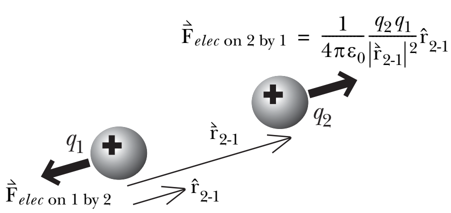


Figure 2.50 The electric force exerted on object 2 by object 1. (The force exerted on object 2 by object 1 has the same magnitude but opposite direction.)

Force laws

THE GRAVITATIONAL FORCE LAW

$$\vec{F}_{grav\text{ on 2 by 1}} = -G\frac{m_2 m_1}{|\vec{r}_{2-1}|^2}\hat{r}_{2-1}$$

$\vec{r}_{2-1}$ points from the center of object 1 to the center of object 2

G is a universal constant: $G = 6.7\times10^{-11}\ \frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}$

Near the Earth's surface, $|\vec{F}_{grav}| = mg$, where $g = +9.8$ N/kg

THE ELECTRIC FORCE LAW (COULOMB'S LAW)

$$\vec{F}_{elec\text{ on 2 by 1}} = \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_1}{|\vec{r}_{2-1}|^2}\hat{r}_{2-1}$$

where $\frac{1}{4\pi\varepsilon_0}$ is a universal constant: $\frac{1}{4\pi\varepsilon_0} = 9\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}$

THE SPRING FORCE LAW (MAGNITUDE)

$$|\vec{F}_{spring}| = k_s|s|$$

s is the stretch: $s = \Delta L = L - L_0$ (relaxed length - new length)

k_s is called the "spring stiffness"

RECIPROCITY

$\vec{F}_{\text{on 1 by 2}} = -\vec{F}_{\text{on 2 by 1}}$ (gravitational and electric forces)

This is also called "Newton's third law of motion."

Additional new concepts and results

Uniform-density spheres act as though all the mass were at the center.

AVERAGE VELOCITY (SPECIAL CASE)

$$\vec{v}_{avg} \approx \frac{(\vec{v}_i + \vec{v}_f)}{2}$$

exactly true *only* if $v << c$ and $\vec{v}$ changes at a constant rate ($\vec{F}_{net}$ constant)

2.18 Review questions

The Momentum Principle

RQ 2.1 At a certain instant, a particle is moving in the $+x$ direction with momentum +10 kg·m/s. During the next 0.1 s, a constant force acts on the particle: $F_x = -6$ N, and $F_y = +3$ N. What is the magnitude of the momentum of the particle at the end of this 0.1 s interval?

RQ 2.2 At $t = 12.0$ seconds an object with mass 2 kg was observed to have a velocity of <10, 35, –8> m/s. At $t = 12.3$ seconds its velocity was <20, 30, 4> m/s. What was the average (vector) net force acting on the object?

RQ 2.3 A proton has mass 1.7×10^{-27} kg. What is the magnitude of the impulse required to increase its speed from $0.990c$ to $0.994c$?

The gravitational force law

RQ 2.4 Calculate the approximate gravitational force exerted by the Earth on a human standing on the Earth's surface. Compare with the approximate gravitational force of a human on another human at a distance of 3 meters. What approximations or simplifying assumptions must you make? (See data on the inside back cover.)

Reciprocity

RQ 2.5 The windshield of a speeding car hits a hovering insect. Compare the magnitude of the force that the car exerts on the bug to the force that the bug exerts on the car. Which is bigger? Compare the magnitude of the change of momentum of the bug to that of the car. Which is bigger? Compare the magnitude of the change of velocity of the bug to that of the car. Which is bigger? Explain briefly. (Note: the interatomic forces between bug and windshield are electric forces.)

The superposition principle

RQ 2.6 In order to pull a sled across a level field at constant velocity you have to exert a constant force. Doesn't this violate Newton's first and second laws of motion, which imply that no force is required to maintain a constant velocity? Explain this seeming contradiction.

2.19 Problems

Problem 2.1 In the space shuttle

A space shuttle is in a circular orbit near the Earth. An astronaut floats in the middle of the shuttle, not touching the walls. On a diagram, draw and label

(a) the momentum $\vec{p}_1$ of the astronaut at this instant;
(b) all of the forces (if any) acting on the astronaut at this instant;
(c) the momentum $\vec{p}_2$ of the astronaut a short time Δt later;
(d) the momentum change (if any) $\Delta\vec{p}$ in this time interval.
(e) Why does the astronaut seem to "float" in the shuttle?

It is ironic that we say the astronaut is "weightless" despite the fact that the only force acting on the astronaut is the astronaut's weight (that is, the gravitational force of the Earth on the astronaut).

Problem 2.2 Two books attract each other

Two copies of this textbook are standing right next to each other on a bookshelf. Make a rough estimate of the magnitude of the gravitational force that the books exert on each other. Explicitly list all quantities that you had to estimate, and all simplifications and approximations you had to make to do this calculation. Compare your result to the gravitational force on the book by the Earth.

Problem 2.3 Compare gravitational and electric forces

Use data from the inside back cover to calculate the gravitational and electric forces two protons exert on each other when they are 1×10^{-10} m apart (about one atomic radius). Which interaction between two protons is stronger, the gravitational attraction or the electric repulsion? If the two protons are at rest, will they begin to move toward each other or away from each other? Note that since both the gravitational and electric force laws depend on the inverse square distance, this comparison holds true at all distances, not just at a distance of 1×10^{-10} m.

Problem 2.4 Crash test

In a crash test, a truck with mass 2200 kg traveling at 25 m/s (about 55 miles per hour) smashes head-on into a concrete wall without rebounding. The front end crumples so much that the truck is 0.8 m shorter than before. What is the approximate magnitude of the force exerted on the truck by the wall? Explain your analysis carefully, and justify your estimates on physical grounds.

Problem 2.5 Ping-pong ball

A ping-pong ball is acted upon by the Earth, air resistance, and a strong wind. Here are the positions of the ball at several times.

Early time interval:

At $t = 12.35$ s, the position was $\langle 3.17, 2.54, -9.38 \rangle$ m
At $t = 12.37$ s, the position was $\langle 3.25, 2.50, -9.40 \rangle$ m

Late time interval:

At $t = 14.35$ s, the position was $\langle 11.25, -1.50, -11.40 \rangle$ m
At $t = 14.37$ s, the position was $\langle 11.27, -1.86, -11.42 \rangle$ m

(a) In the early time interval, from $t = 12.35$ s to $t = 12.37$ s, what was the average momentum of the ball? The mass of the ping-pong ball is 2.7 grams (2.7×10^{-3} kg). Express your result as a vector.

(b) In the late time interval, from $t = 14.35$ s to $t = 14.37$ s, what was the average momentum of the ball? Express your result as a vector.

(c) In the time interval from $t = 12.35$ s (the start of the early time interval) to $t = 14.35$ s (the start of the late time interval), what was the average net force acting on the ball? Express your result as a vector.

Problem 2.6 Kick a basketball

A 0.6 kg basketball is rolling by you at 3.5 m/s. As it goes by, you give it a kick perpendicular to its path (Figure 2.51). Your foot is in contact with the ball for 0.002 s. The ball eventually rolls at a 20° angle from its original direction. The overhead view is approximately to scale. The arrow represents the force your toe applies briefly to the basketball.

(a) In the diagram, which letter corresponds to the correct overhead view of the ball's path?

(b) Determine the magnitude of the average force you applied to the ball.

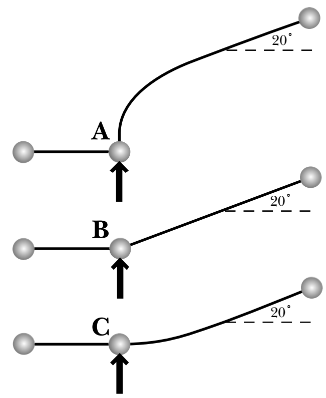


Figure 2.51 Kick a basketball (Problem 2.6).

Problem 2.7 Spacecraft and asteroid

At $t = 532.0$ s after midnight a spacecraft of mass 1400 kg is located at position $\langle 3\times10^5, 7\times10^5, -4\times10^5\rangle$ m, and an asteroid of mass 7×10^{15} kg is located at position $\langle 9\times10^5, -3\times10^5, -12\times10^5\rangle$ m. There are no other objects nearby.

(a) Calculate the (vector) force acting on the spacecraft.

(b) At $t = 532.0$ s the spacecraft's momentum was $\vec{p}_i$, and at the later time $t = 538.0$ s its momentum was $\vec{p}_f$, Calculate the (vector) change of momentum $\vec{p}_f - \vec{p}_i$.

Problem 2.8 Proton and HCl molecule

A proton interacts electrically with a neutral HCl molecule located at the origin. At a certain time t, the proton's position is $\langle 1.6\times10^{-9}, 0, 0\rangle$ m and the proton's velocity is $\langle 3200, 800, 0\rangle$ m/s. The force exerted on the proton by the HCl molecule is $\langle -1.12\times10^{-11}, 0, 0\rangle$ N. At a time $t + (2\times10^{-14}$ s$)$, what is the approximate velocity of the proton?

Problem 2.9 A free throw in basketball

Determine two different possible ways for a player to make a free throw in basketball. In both cases give the initial speed, initial angle, and initial height of the basketball. The rim of the basket is 10 feet (3.0 m) above the floor. It is 14 feet (4.3 m) along the floor from the free-throw line to a point directly below the center of the basket.

Problem 2.10 A basketball pass

You have probably seen a basketball player throw the ball to a teammate at the other end of the court, 30 m away. Estimate a reasonable initial angle for such a throw, and then determine the corresponding initial speed. For your chosen angle, how long does it take for the basketball to go the length of the court? What is the highest point along the trajectory, relative to the thrower's hand?

Problem 2.11 The case of the falling flower pot

You are a detective investigating why someone was hit on the head by a falling flowerpot. One piece of evidence is a home video taken in a 4th-floor apartment, which happens to show the flowerpot falling past a tall window. Inspection of individual frames of the video shows that in a span of 6 frames the flowerpot falls a distance that corresponds to 0.85 of the window height seen in the video (note: standard video runs at a rate of 30 frames per second). You visit the apartment and measure the window to be 2.2 m high. What can you conclude? Under what assumptions? Give as much detail as you can.

Problem 2.12 Tennis ball hits wall

A tennis ball has a mass of 0.057 kg. A professional tennis player hits the ball hard enough to give it a speed of 50 m/s (about 120 miles per hour). The ball hits a wall and bounces back with almost the same speed (50 m/s). As indicated in Figure 2.52, high-speed photography shows that the ball is

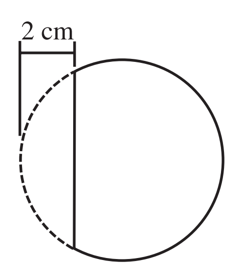


Figure 2.52 A high-speed tennis ball deforms when it hits a wall (Problem 2.12).

crushed 2 cm (0.02 m) at the instant when its speed is momentarily zero, before rebounding.

Making the very rough approximation that the large force that the wall exerts on the ball is approximately constant during contact, determine the approximate magnitude of this force. Hint: Think about the approximate amount of time it takes for the ball to come momentarily to rest. (For comparison note that the gravitational force on the ball is quite small, only about (0.057 kg)(9.8 N/kg) ≈ 0.6 N. A force of 5 N force is approximately the same as a force of one pound.)

Problem 2.13 Mars probe

A small space probe, of mass 240 kg, is launched from a spacecraft near Mars. It travels toward the surface of Mars, where it will land. At a time 20.7 seconds after it is launched, the probe is at the location $\langle 4.30\times10^3, 8.70\times10^2, 0\rangle$ m, and at this same time its momentum is $\langle 4.40\times10^4, -7.60\times10^3, 0\rangle$ kg · m/s. At this instant, the net force on the probe due to the gravitational pull of Mars plus the air resistance acting on the probe is $\langle -7\times10^3, -9.2\times10^2, 0\rangle$ N.

(a) Assuming that the net force on the probe is approximately constant over this time interval, what is the momentum of the probe 20.9 seconds after it is launched?

(b,) What is the location of the probe 20.9 seconds after launch?

Problem 2.14 Spacecraft navigation

Suppose you are navigating a spacecraft far from other objects. The mass of the spacecraft is 1.5×10^5 kg (about 150 tons). The rocket engines are shut off, and you're coasting along with a constant velocity of $\langle 0, 20, 0\rangle$ km/s. As you pass the location $\langle 12, 15, 0\rangle$ km you briefly fire side thruster rockets, so that your spacecraft experiences a net force of $\langle 6\times10^4, 0, 0\rangle$ N for 3.4 s. The ejected gases have a mass that is small compared to the mass of the spacecraft. You then continue coasting with the rocket engines turned off. Where are you an hour later? Also, what approximations or simplifying assumptions did you have to make in your analysis? Think about the choice of system: what are the surroundings that exert external forces on your system?

Problem 2.15 Electron motion in a CRT

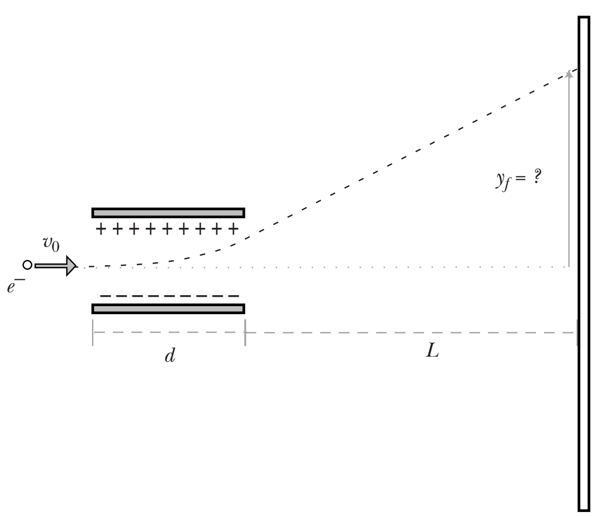


Figure 2.53 A cathode ray tube (Problem 2.15).

In a cathode ray tube (CRT) used in oscilloscopes and television sets, a beam of electrons is steered to different places on a phosphor screen, which glows at locations hit by electrons (Figure 2.53). The CRT is evacuated, so there are few gas molecules present for the electrons to run into. Electric forces are used to accelerate electrons of mass m to a speed $v_0 << c$, after which they pass between positively and negatively charged metal plates which deflect the electron in the vertical direction (upward in Figure 2.53, or downward if the sign of the charges on the plates is reversed).

While an electron is between the plates, it experiences a uniform vertical force F, but when the electron is outside the plates there is negligible force on it. The gravitational force on the electron is negligibly small compared to the electric force in this situation. The length of the metal plates is d, and the phosphor screen is a distance L from the metal plates. Where does the electron hit the screen? (That is, what is y_f?)

Problem 2.16 Star and planet

A star of mass 7×10^{30} kg is located at $\langle 5\times10^{12}, 2\times10^{12}, 0\rangle$ m. A planet of mass 3×10^{24} kg is located at $\langle 3\times10^{12}, 4\times10^{12}, 0\rangle$ m and is moving with a velocity of $\langle 0.3\times10^{4}, 1.5\times10^{4}, 0\rangle$ m/s.

(a) At a time 1×10^{6} seconds later, what is the new velocity of the planet?

(b) Where is the planet at this later time?

(c) Explain briefly why the procedures you followed in parts (a) and (b) were able to produce usable results but wouldn't work if the later time had been 1×10^{9} seconds instead of 1×10^{6} seconds after the initial time. Explain briefly how could you use a computer to get around this difficulty.

Problem 2.17 Two stars

At $t = 0$ a star of mass 4×10^{30} kg has velocity $\langle 7\times10^{4}, 6\times10^{4}, -8\times10^{4}\rangle$ m/s and is located at $\langle 2.00\times10^{12}, -5.00\times10^{12}, 4.00\times10^{12}\rangle$ m relative to the center of a cluster of stars. There is only one nearby star that exerts a significant force on the first star. The mass of the second star is 3×10^{30} kg, its velocity is $\langle 2\times10^{4}, -1\times10^{4}, 9\times10^{4}\rangle$ m/s, and this second star is located at $\langle 2.03\times10^{12}, -4.94\times10^{12}, 3.95\times10^{12}\rangle$ m relative to the center of the cluster of stars.

(a) At $t = 1\times10^{5}$ s, what is the approximate momentum of the first star?

(b) Discuss briefly some ways in which your result for (a) is approximate, not exact.

(c) At $t = 1\times10^{5}$ s, what is the approximate position of the first star?

(d) Discuss briefly some ways in which your result for (b) is approximate, not exact.

Problem 2.18 The SLAC two-mile accelerator

SLAC, the Stanford Linear Accelerator Center, located at Stanford University in Palo Alto, California, accelerates electrons through a vacuum tube two miles long (it can be seen from an overpass of the Junipero Serra freeway that goes right over the accelerator). Electrons which are initially at rest are subjected to a continuous force of 2×10^{-12} newton along the entire length of two miles (one mile is 1.6 kilometers) and reach speeds very near the speed of light.

(a) Determine how much time is required to increase the electrons' speed from $0.93c$ to $0.99c$. (That is, the quantity $|\vec{v}|/c$ increases from 0.93 to 0.99.)

(b) Approximately how far does the electron go in this time? What is approximate about your result?

Problem 2.19 Determining the mass of an asteroid

In June 1997 the NEAR spacecraft ("Near Earth Asteroid Rendezvous"; see http://near.jhuapl.edu/), on its way to photograph the asteroid Eros, passed within 1200 km of asteroid Mathilde at a speed of 10 km/s relative to the asteroid (Figure 2.54). From photos transmitted by the 805 kg spacecraft, Mathilde's size was known to be about 70 km by 50 km by 50 km. It is presumably made of rock. Rocks on Earth have a density of about 3000 kg/m^3 (3 grams/cm^3).

(a) Make a rough diagram to show qualitatively the effect on the spacecraft of this encounter with Mathilde. Explain your reasoning.

(b) Make a very rough estimate of the change in momentum of the spacecraft that would result from encountering Mathilde. Explain how you made your estimate.

(c) Using your result from part (b), make a rough estimate of how far off course the spacecraft would be, one day after the encounter.

(d) From actual observations of the location of the spacecraft one day after encountering Mathilde, scientists concluded that Mathilde is a loose ar-

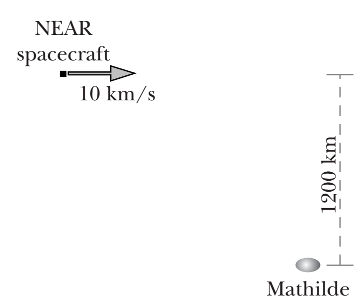


Figure 2.54 The NEAR spacecraft passes the asteroid Mathilde (Problem 2.19).

rangement of rocks, with lots of empty space inside. What was it about the observations that must have led them to this conclusion?

Experimental background: The position was tracked by very accurate measurements of the time that it takes for a radio signal to go from Earth to the spacecraft followed immediately by a radio response from the spacecraft being sent back to Earth. Radio signals, like light, travel at a speed of 3×10^8 m/s, so the time measurements had to be accurate to a few nanoseconds ($1\ \text{ns} = 10^{-9}\ \text{s}$).

The following problems are intended to introduce you to using a computer to model matter, interactions, and motion. Some parts of these problems can be done with almost any tool (spreadsheet, math package, etc.). Other parts are most easily done with a programming language, for which we recommend the free 3D programming language VPython (http://vpython.org). Your instructor will introduce you to an available computational tool and assign problems, or parts of problems, that can be addressed using the chosen tool.

Problem 2.20 Planetary orbits

In this problem you will study the motion of a planet around a star. To start with a somewhat familiar situation, you will begin by modeling the motion of our Earth around our Sun. Write answers to questions either as comments in your program or on paper, as specified by your instructor.

Planning

(a) The Earth goes around the Sun in a nearly circular orbit, taking one year to go around. Using data on the inside back cover, what initial speed should you give the Earth in a computer model, so that a circular orbit should result? (If your computer model does produce a circular orbit, you have a strong indication that your program is working properly.)

(b) Estimate an appropriate value for Δt to use in your computer model. Remember that t is in seconds, and consider your answer to part (a) in making this estimate. If your Δt is too small, your calculation will require many tedious steps. Explain briefly how you decided on an appropriate step size.

Circular orbit

(c) Starting with speed calculated in (a), and with the initial velocity perpendicular to a line connecting the Sun and the Earth, calculate and display the trajectory of the Earth. Display the whole trail, so you can see whether you have a closed orbit (that is, whether the Earth returns to its starting point each time around). Include the Sun's position $\vec{r}_s = \langle x_s, y_s, z_s \rangle$ in your computations, even if you set its coordinates to zero. This will make it easier to modify the computation to let the Sun move (Problem 2.2 Binary stars).

Computational accuracy

(d) As a check on your computation, is the orbit a circle as expected? Run the calculation until the accumulated time t is the length of a year: does this take the Earth around the orbit once, as expected? What is the largest step size you can use that still gives a circular orbit and the correct length of a year? What happens if you use a much larger step size?

Noncircular trajectories

(e) Set the initial speed to 1.2 times the Earth's actual speed. Make sure the step size is small enough that the orbit is nearly unaffected if you cut the step size. What kind of orbit do you get? What step size do you need to use to get results you can trust? What happens if you use a much larger step size? Produce at least one other qualitatively different noncircular trajectory for the Earth. What difficulties would humans have in surviving on the Earth if it had a highly noncircular orbit?

Force and momentum

(f) Choose initial conditions that give a noncircular orbit. Continuously display a vector showing the momentum of the Earth (with its tail at the Earth's position), and a different colored vector showing the force on the Earth by the Sun (with its tail at the Earth's position). You must scale the vectors appropriately to fit into the scene. A way to figure out what scale factor to use is to print the numerical values of momentum and force, and compare with the scale of the scene. For example, if the width of the scene is W meters, and the force vector has a typical magnitude F (in newtons), you might scale the force vector by a factor $0.1\,W/F$, which would make the length of the vector be one-tenth the width of the scene. Is the force in the same direction as the momentum? How does the momentum depend on the distance from the star?

Problem 2.21 Binary stars

About a half of the visible "stars" are actually systems consisting of two stars orbiting each other, called "binary stars." In your computer model of the Earth and Sun (Problem 2.20 Planetary orbits), replace the Earth with a star whose mass is half the mass of our Sun, and take into account the gravitational effects that the second star has on the Sun.

(a) Give the second star the speed of the actual Earth, and give the Sun zero initial momentum. What happens? Try a variety of other initial conditions. What kinds of orbits do you find?

(b) One of the things to try is to give the Sun the same magnitude of momentum as the second star, but in the opposite direction. If the initial momentum of the second star is in the $+y$ direction, give the Sun the same magnitude of momentum but in the $-y$ direction, so that the total momentum of the binary star system is initially zero, but the stars are not headed toward each other. What is special about the motion you observe in this case?

Problem 2.22 Other force laws

Modify your orbit computation to use a different force law, such as a force that is proportional to $1/r$ or $1/r^3$, or a constant force, or a force proportional to r^2 (this represents the force of a spring whose relaxed length is nearly zero). How do orbits with these force laws differ from the circles and ellipses that result from a $1/r^2$ law? If you want to keep the magnitude of the force roughly the same as before, you will need to adjust the force constant G.

Problem 2.23 The effect of the Moon and Venus on the Earth

In Problem 2.20 you analyzed a simple model of the Earth orbiting the Sun, in which there were no other planets or moons. Venus and the Earth have similar size and mass. At its closest approach to the Earth, Venus is about 40 million kilometers away (4×10^{10} m). The Moon's mass is about 7×10^{22} kg, and the distance from Earth to Moon is about 4×10^{8} m (400,000 km, center to center).

(a) Calculate the ratio of the gravitational forces on the Earth exerted by Venus and the Sun. Is it a good approximation to ignore the effect of Venus when modeling the motion of the Earth around the Sun?

(b) Calculate the ratio of the gravitational forces on the Earth exerted by the Moon and the Sun. Is it a good approximation to ignore the effect of the Moon when modeling the motion of the Earth around the Sun?

Problem 2.24 The three-body problem

Carry out a numerical integration of the motion of a three-body gravitational system and plot the trajectory, leaving trails behind the objects. Calculate all of the forces before using these forces to update the momenta and positions of the objects. Otherwise the calculations of gravitational forces would mix positions corresponding to different times.

Try different initial positions and initial momenta. Find at least one set of initial conditions that produces a long-lasting orbit, one set of initial conditions that results in a collision with a massive object, and one set of initial conditions that allows one of the objects to wander off without returning. Report the masses and initial conditions that you used.

Problem 2.25 The Ranger 7 mission to the Moon

The first U.S. spacecraft to photograph the Moon close up was the unmanned "Ranger 7" photographic mission in 1964. The spacecraft, shown in the illustration at the right (NASA photograph), contained television cameras that transmitted close-up pictures of the Moon back to Earth as the spacecraft approached the Moon. The spacecraft did not have retro-rockets to slow itself down, and it eventually simply crashed onto the Moon's surface, transmitting its last photos immediately before impact.

Figure 2.55 The first Ranger 7 photo of the Moon (NASA photograph).

Figure 2.55 is the first image of the Moon taken by a U.S. spacecraft, Ranger 7, on July 31, 1964, about 17 minutes before impacting the lunar surface. The large crater at center right is Alphonsus (108 km diameter); above it (and to the right) is Ptolemaeus and below it is Arzachel. The Ranger 7 impact site is off the frame, to the left of the upper left corner.

You can find out more about the actual Ranger lunar missions at

http://nssdc.gsfc.nasa.gov/planetary/lunar/ranger.html

To send a spacecraft to the Moon, we put it on top of a large rocket containing lots of rocket fuel and fire it upward. At first the huge ship moves quite slowly, but the speed increases rapidly. When the "first-stage" portion of the rocket has exhausted its fuel and is empty, it is discarded and falls back to Earth. By discarding an empty rocket stage we decrease the amount of mass that must be accelerated to even higher speeds. There may be several stages that operate for a while and then are discarded before the spacecraft has risen above most of Earth's atmosphere (about 50 km, say, above the Earth), and has acquired a high speed. At that point all the fuel available for this mission has been used up, and the spacecraft simply coasts toward the Moon through the vacuum of space.

We will model the Ranger 7 mission. Starting 50 km above the Earth's surface (5×10^4 m), a spacecraft coasts toward the Moon with an initial speed of about 10^4 m/s. Here are data we will need:

mass of spacecraft = 173 kg   mass of Earth $\approx 6\times10^{24}$ kg

mass of Moon $\approx 7\times10^{22}$ kg   radius of Moon = 1.75×10^6 m

distance from Earth to Moon $\approx 4\times10^8$ m (400,000 km, center to center)

We're going to ignore the Sun in a simplified model even though it exerts a sizable gravitational force. We're expecting the Moon mission to take only a few days, during which time the Earth (and Moon) move in a nearly straight line with respect to the Sun, because it takes 365 days to go all the way around the Sun. We take a reference frame fixed to the Earth as representing (approximately) an inertial frame of reference with respect to which we can use the Momentum Principle.

For a simple model, make the Earth and Moon be fixed in space during the mission. Factors that would certainly influence the path of the spacecraft include the motion of the Moon around the Earth, and the motion of the Earth around the Sun. In addition, the Sun and other planets exert gravitational forces on the spacecraft. As a separate project you might like to include some of these additional factors.

(a) Compute the path of the spacecraft, and display it either with a graph or with an animated image. Here, and in remaining parts of the problem, report the step size Δt that gives accurate results (that is, cutting this step size has little effect on the results).

(b) By trying various initial speeds, determine the approximate *minimum* launch speed needed to reach the Moon, to two significant figures (this is the speed that the spacecraft obtains from the multistage rocket, at the time of release above the Earth's atmosphere). What happens if the launch speed is less than this minimum value? (Be sure to check the step size issue.)

(c) Use a launch speed 10% larger than the approximate minimum value found in part (b). How long does it take to go to the Moon, in hours or days? (Be sure to check the step size issue.)

(e) What is the "impact speed" of the spacecraft (its speed just before it hits the Moon's surface)? Make sure that your spacecraft crashes on the surface of the Moon, not at the Moon's center! (Be sure to check the step size issue.)

You may have noticed that you don't actually need to know the mass *m* of the spacecraft in order to carry out the computation. The gravitational force is proportional to *m*, and the momentum is also proportional to *m*, so *m* cancels. However, nongravitational forces such as electric forces are not proportional to the mass, and there is no cancellation in that case. We kept the mass *m* in the analysis in order to illustrate a general technique for predicting motion, no matter what kind of force, gravitational or not.

Problem 2.26 The effect of Venus on the Moon voyage

In the Moon voyage analysis, you used a simplified model in which you neglected among other things the effect of Venus. An important aspect of physical modeling is making estimates of how large the neglected effects might be. If we take Venus into account, make a rough estimate of whether the spacecraft will miss the Moon entirely. How large a sideways deflection of the crash site will there be? Explain your reasoning and approximations.

Venus and the Earth have similar size and mass. At its closest approach to the Earth, Venus is about 40 million kilometers away (4×10^{10} m). In the real world, Venus would attract the Earth and the Moon as well as the spacecraft, but to get an idea of the size of the effects, pretend that the Earth, the Moon, and Venus are all fixed in position (Figure 2.56) and just investigate Venus's effect on the spacecraft.

Figure 2.56 The relative positions of Venus, Earth, and Moon in Problem 2.26.

2.20 Answers to exercises

2.1 (page 55) 0.275 N

2.2 (page 55) 3571 N/m

2.3 (page 55) 71.4 N

2.4 (page 56) $\langle -60, -24, 96 \rangle$ N·s

2.5 (page 58) $\langle -0.03, -005, 0.02 \rangle$ N

2.6 (page 58) **Earth,** water, and air exert forces; net force is zero

2.7 (page 58) zero

2.8 (page 59) glider; track, spring, Earth, air

2.9 (page 63) $\vec{p}_f = \langle 10, 0, 11 \rangle$ kg·m/s

2.10 (page 63) $\Delta\vec{p} = \vec{p}_f - \vec{p}_i = \langle -15000, 0, 3000 \rangle$ kg·m/s = impulse by ground

2.11 (page 63) $\vec{F}_{\text{net}} = \langle -5000, 0, 1000 \rangle$ N

2.12 (page 63) $\vec{F}_{\text{net}} = \langle -1\times10^4, 0, 0 \rangle$ N

2.13 (page 63) $\vec{p}_f = \langle 9.4, 0.3, 0 \rangle$ kg·m/s

2.14 (page 67) (a) $\langle -10, 7.12, -5 \rangle$ m/s

(b) $\langle 3, 6.036, -8 \rangle$ m

(c) 8.6 m

(d) 1.33 s

(e) 2.66 s

(f) $\langle -17.5, 0, -18.3 \rangle$ m

2.15 (page 67) $y_i = h$, $v_{yi} = 0$, $y_f = 0$, $\Delta t = \sqrt{2h/g}$, $v_{yf} = \sqrt{2gh}$

Same results if mass is changed.

2.16 (page 68) time in air is 1.77 s; horizontal distance 8.8 m; 3.8 m high

2.17 (page 72) $y = 0.108$ m, $p_y = 0.0365$ kg·m/s

2.18 (page 82) 3/16 times as large

2.19 (page 82) 2.5×10^{-10} N, 2.5×10^{-10} N

2.20 (page 82) 6×10^{24} kg

2.21 (page 86) (a) $\langle 2.8\times10^8, 0, -2.8\times10^8 \rangle$ m

(b) 3.96×10^8 m

(c) $\langle 0.707, 0, -0.707 \rangle$ (no units; dimensionless

(d) $\langle -1.27\times10^{20}, 0, 1.27\times10^{20} \rangle$ N

2.22 (page 90) about a day or less; a small portion of the circle

2.23 (page 90) about 0.02 s; a small fraction of the round-trip time

2.25 (page 91) 3.2×10^4 m; note that Mt. Everest is about 8×10^3 m tall

2.26 (page 92) 2.3×10^{-8} N