PHYS 501: Mathematical Physics I

Fall 2018, Homework #6

(Due December 7, 2018)

1. Poisson's equation (in <u>three</u> dimensions) is

$$\nabla^2\phi = 4\pi G\rho.$$

(a) Let $\tilde{\phi}(\mathbf{k})$ and $\tilde{\rho}(\mathbf{k})$ be the Fourier transforms of $\phi(\mathbf{x})$ and $\rho(\mathbf{x})$, respectively. Show that

$$\tilde{\phi} = -\frac{4\pi G\tilde{\rho}}{k^2},$$

and hence write down an integral expression for $\phi(\mathbf{x})$.

(b) For a point mass at the origin, $\rho(\mathbf{x}) = m\delta(\mathbf{x})$. Use the result of part (a) to determine the solution for ϕ.

2. Find the Green's function $G(x, x')$ for the equation

$$\frac{d^2y}{dx^2} - k^2 y = f(x)\,,$$

for $0 \le x \le L$, with $y(0) = y(L) = 0$. (Find G by solving the differential equation, not just as a formal sum over eigenfunctions!)

3. By first considering the behavior of the fundamental solution in the vicinity of $\mathbf{x} = \mathbf{x}'$, show that the Green's function $G(\mathbf{x}, \mathbf{x}')$ for the three-dimensional Helmholtz equation

$$\nabla^2 u + k^2 u = 0\,,$$

with the boundary condition that $u(\mathbf{x})e^{-i\omega t}$ represents outgoing waves at infinity, is

$$G(\mathbf{x}, \mathbf{x}') = -\frac{e^{ikr}}{4\pi r}\,,$$

where $r = |\mathbf{x} - \mathbf{x}'|$.

4. In using the method of images to find the Dirichlet Green's function for Poisson's equation inside a sphere of radius a, it can be shown that the image of a point $\mathbf{x}$ within the sphere is $\mathbf{x}_1 = \alpha\mathbf{x}$, with strength β, so that the Green's function is

$$G(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi|\mathbf{x}' - \mathbf{x}|} + \frac{\beta}{4\pi|\mathbf{x}' - \mathbf{x}_1|}.$$

(a) The boundary condition on G is that $G(\mathbf{x}, \mathbf{x}') = 0$ if $\mathbf{x}'$ lies on the surface of the sphere. By applying this condition to the two points where the diameter through $\mathbf{x}$ intersects the surface of the sphere, show that $\beta = a/r$ and $\alpha = \beta^2$, where $r = |\mathbf{x}|$.

(b) Hence derive an expression for the solution $u(r, \theta, \phi)$ to Laplace's equation $\nabla^2 u = 0$ inside the sphere, subject to the boundary condition $u(a, \theta, \phi) = f(\theta, \phi)$.

(c) Compare this form of the solution with the series solution obtained by separation of variables within the sphere $r < a$.

5. As discussed in class, the Green's function solution to the inhomogeneous wave equation

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = f(\mathbf{x}, t)$$

is the "retarded potential" solution

$$\phi(\mathbf{x}, t) = -\frac{c}{4\pi}\int d^3\mathbf{x}'\, dt'\; f(\mathbf{x}', t')\; \frac{\delta[|\mathbf{x} - \mathbf{x}'| - c(t - t')]}{|\mathbf{x} - \mathbf{x}'|},$$

where $\frac{1}{c}|\mathbf{x} - \mathbf{x}'|$ is the light travel time from the source point $\mathbf{x}'$ to the field point $\mathbf{x}$.

Now suppose that the field is due to a moving point source of unit magnitude, so

$$f(\mathbf{x}', t') = \delta[\mathbf{x}' - \boldsymbol{\xi}(t')],$$

that is, the trajectory of the source is given by $\mathbf{x}' = \boldsymbol{\xi}(t')$. Use the above equation to find a closed-form expression (no integrals!) for the potential $\phi(\mathbf{x}, t)$.