PHYS 501: Mathematical Physics I

Fall 2018, Homework #5

(Due November 29, 2018)

1. (a) Fill in the blanks, where $u(x, y)$ and $v(x, y)$ are the real and imaginary parts, respectively, of the analytic function $w(z)$:

   (i) $u(x, y) = e^{2x} \cos 2y, \quad v(x, y) = ?, \quad w(z) = ?$

   (ii) $u(x, y) = ?, \quad v(x, y) = y(3x^2 - y^2 - 2), \quad w(z) = ?$

   (iii) $u(x, y) = ?, \quad v(x, y) = ?, \quad w(z) = \tan^{-1} z.$

   (b) Find *all* Laurent or Taylor expansions of the function

   $$f(z) = \frac{z}{z^2 + 1}$$

   about the point $z = 2i$, i.e. expand the function as a series of the form

   $$f(z) = \sum_{n=-\infty}^{\infty} c_n \, s^n.$$

   where $s = z - 2i$. Note that there are several different regions in which different expansions apply.

2. Evaluate the following integrals using the residue theorem:

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^3}$ (b) $\displaystyle\int_{0}^{\infty} \frac{x \sin kx \; dx}{x^2 + 1}$

   (c) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ikx} \; dx}{(x^2 + a^2)(x^2 + b^2)}$ (d) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 e^x \; dx}{1 + e^{2x}}$

   where a and b are nonzero and $a \neq b$. In each case, sketch the contour you choose and clearly quote all theorems used in the derivation of your results.

3. Use contour integration to find the inverse Fourier transform $f(t)$ of the function

   $$F(\omega) = \sqrt{\frac{2}{\pi}} \, \frac{\sin \omega a}{\omega}$$

   (where $a > 0$), for all values of t. (Use the "$(2\pi)^{-1/2}$" version of the transform.) Recall that F was obtained as the Fourier transform of a step function with a discontinuity at $|t| = a$. What is the value of $f(a)$? (Determine $f(a)$ from the integral—don't appeal to the general properties of Fourier transforms!)

4. Find the (3-D) Fourier transform of the wave function for a $2p$ electron in a hydrogen atom:

   $$\psi(\mathbf{x}) = (32\pi a_0^5)^{-1/2} \, z e^{-r/2a_0} \, ,$$

   where $a_0 = \hbar^2/me^2$ is the Bohr radius, r is radius, and z is a rectangular coordinate.