PHYS 501: Mathematical Physics I

Fall 2018

Solutions to Homework #5

1. (a) (i) The Cauchy–Riemann equations imply $\partial u/\partial x = \partial v/\partial y = 2e^{2x}\cos 2y$ and $\partial u/\partial y = -\partial v/\partial x = -2e^{2x}\sin 2y$, so $v = e^{2x}\sin 2y + \text{constant}$, so $w(z) = e^{2z} + c_1$, where c_1 is pure imaginary.

   (ii) Similar reasoning gives $w(z) = z^3 - 2z + c_2$, where c_2 is real.

   (iii) Inverting the relation $z = \tan w = \frac{\sin w}{\cos w} = -i\frac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}}$, we obtain

$$\begin{aligned} w(z) &= \tfrac{1}{2}i\,\log\left(\frac{1-iz}{1+iz}\right) = -\tfrac{1}{2}\arg\left[\frac{1-iz}{1+iz}\right] + \tfrac{1}{2}i\,\log\left|\frac{1-iz}{1+iz}\right| \\ &= -\tfrac{1}{2}\left[\arg(1-iz) - \arg(1+iz)\right] + \tfrac{1}{4}i\,\log\left[\frac{x^2+(1+y)^2}{x^2+(1-y)^2}\right], \end{aligned}$$

   so

$$\begin{aligned} u(x,y) &= \tfrac{1}{2}\left[\tan^{-1}\frac{x}{1+y} + \tan^{-1}\frac{x}{1-y}\right] = \tfrac{1}{2}\tan^{-1}\frac{2x}{1-x^2-y^2}, \\ v(x,y) &= \tfrac{1}{4}\log\left[\frac{x^2+(1+y)^2}{x^2+(1-y)^2}\right]. \end{aligned}$$

   (b) The function has poles at $z = \pm i$, so we expect three different expansions in $s = z - 2i$, for $|s| < 1$ (Taylor series), $1 < |s| < 3$ (mixed Laurent series), and $|s| > 3$ (pure Laurent series). Write $f(z) = \frac{1}{2}\left(\frac{1}{z-i} + \frac{1}{z+i}\right) = \frac{1}{2}\left(\frac{1}{s+i} + \frac{1}{s+3i}\right)$ and construct Taylor or Laurent series in s using appropriate binomial expansions in each is range. For $|s| < 1$ expand both fractions as Taylor series in is. For $1 < |s| < 3$, expand the second fraction as a Taylor series, and rewrite the first as $\frac{1}{2s}\left(\frac{1}{1+i/s}\right)$, where $|i/s| < 1$, and expand as a series in $(is)^{-1}$. For $|s| > 3$ expand both fractions as series in $(is)^{-1}$.

   The Taylor and Laurent series for $f_1 = \frac{1}{2}\left(\frac{1}{s+i}\right)$ are $f_1 = -\frac{i}{2}\sum_{n=0}^{\infty}(is)^n$ $(|s| < 1)$ and $f_1 = \frac{1}{2s}\sum_{n=0}^{\infty}(is)^{-n}$ $(|s| > 1)$. For $f_2 = \frac{1}{2}\left(\frac{1}{s+3i}\right)$, we have $f_2 = -\frac{i}{6}\sum_{n=0}^{\infty}(is/3)^n$ $(|s| < 3)$ and $f_2 = \frac{1}{2s}\sum_{n=0}^{\infty}(is/3)^{-n}$ $(|s| > 3)$. Hence the possible expansions are

$$|s| < 1: \qquad f(z) = \frac{-i}{2}\sum_{n=0}^{\infty}\left(1 + 3^{-n-1}\right)(is)^n$$

$$1 < |s| < 3: \qquad f(z) = \frac{1}{2s}\sum_{n=0}^{\infty}(is)^{-n} - \frac{i}{6}\sum_{n=0}^{\infty}\left(\frac{is}{3}\right)^n$$

$$|s| > 3: \qquad f(z) = \frac{1}{2s}\sum_{n=0}^{\infty}\left(1 + 3^n\right)(is)^{-n}$$

   Note that the poles of f lie on the circles where the various expansions diverge.

2. (a) Complete the contour with a large semicircle C_R of radius R in Im $z > 0$, argue that the integral along C_R is less than $\pi R/R^6 \to 0$ as $R \to \infty$, and evaluate the residue at the pole (of order 3) $z = i$ as $-3i/16$ to find $I = 3\pi/8$.

(b) Write $I = \frac{1}{2}\,\mathrm{Im} \int_{-\infty}^{\infty} \frac{z e^{ikz}}{z^2+1}\, dz$. For $k > 0$ complete the contour as in part (a), use Jordan's lemma to show that the integral around C_R goes to zero as $R \to \infty$, and evaluate the residue $(e^{-k}/2)$ at $z = i$ to find $I = \frac{\pi}{2} e^{-k}$. For $k < 0$ complete the contour in the negative Imaginary half plane, again use Jordan's lemma to show that the integral around C_R goes to zero as $R \to \infty$, and evaluate the residue $(e^{k}/2)$ at $z = -i$ to find $I = -\frac{\pi}{2} e^{k}$. Hence, in either case, $I = \mathrm{sign}(k) \frac{\pi}{2} e^{-|k|}$

(c) Write $I = \int_{-\infty}^{\infty} \frac{e^{ikz}\, dz}{(z^2+a^2)(z^2+b^2)}$, complete the contour as in part (b), and as usual the integral around C_R goes to zero as $R \to \infty$. For $k > 0$ use the semicircle in the upper imaginary half plane; for $k < 0$ use the lower imaginary half plane.

There are four poles, at $z = \pm ia$ and $\pm ib$, but only two of them lie inside either contour. The residue at ia is $\frac{ie^{-ka}}{2a(a^2-b^2)}$, with similar expressions for the other residues. For any choice of a, b, or k, the solution may be written as

$$I = \frac{\pi}{b^2 - a^2} \left[\frac{e^{-|ka|}}{|a|} - \frac{e^{-|kb|}}{|b|} \right].$$

(d) The integral is $I = \int_{-\infty}^{\infty} \frac{z^2}{2\cosh z}\, dz$. In this case, the usual trick of completing the contour with a large semicircle will not work, as the integrand has an infinite number of poles along the imaginary axis and the integral along the large semicircle does not go to zero. Instead, the contour to use is a rectangle having corners $-R$, R, $R + \pi i$, and $-R + \pi i$. This exploits the periodicity of $\cosh z$ in the imaginary direction. The sides parallel to the imaginary axis do not contribute as $R \to \infty$ and, using $\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$, it follows that the integral along the top side is

$$\begin{aligned} \int_{\infty}^{-\infty} \frac{(x+i\pi)^2}{2\cosh(x+i\pi)}\, dx &= \int_{-\infty}^{\infty} \frac{(x^2 - 2i\pi x - \pi^2)}{2\cosh x}\, dx \\ &= I - \tfrac{1}{2}\pi^2 \int_{-\infty}^{\infty} \frac{dz}{\cosh z}\,. \end{aligned}$$

The second integral may be shown (by a repetition of the same approach, as done in class) to be π. Thus we have

$$2I - \pi^3/2 = 2\pi i \times \mathrm{Res}(\pi i/2) \quad \Rightarrow \quad I = \pi^3/8\,.$$

3. Write

$$\sin \omega a = \frac{e^{i\omega a} - e^{-i\omega a}}{2i},$$

splitting the inverse transform into two integrals,

$$\begin{aligned} I_1 &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega\, \frac{e^{i\omega(t+a)}}{\omega} \\ I_2 &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega\, \frac{e^{i\omega(t-a)}}{\omega}. \end{aligned}$$

Both integrals have singularities at $\omega = 0$, on the integration path. We avoid them with small semicircular contours bypassing the singularity above, adding a contribution of $-\pi i$ times the relevant residue at $\omega = 0$. In all cases, the residue is $1/2i$, leading to a contribution of $-\frac{1}{2}$.

We close the contours with large semicircles at $|\omega| = \infty$. For integral I_1, Jordan's lemma implies that we close the contour in $\text{Im}(\omega) > 0$ for $t < a$, and in $\text{Im}(\omega) < 0$ for $t > a$. Thus, for $t < a$ we have $I_1 - \frac{1}{2} = 0$, since no poles are enclosed by the chosen contour, so $I_1 = \frac{1}{2}$. For $t > a$, we have $I_1 - \frac{1}{2} = -1$ (negative since the contour is traversed in the clockwise direction), so $I_1 = -\frac{1}{2}$. Similarly, we have $I_2 = \frac{1}{2}$ for $t < -a$, $I_2 = -\frac{1}{2}$ for $t > -a$.

Combining these results, we have

$$I = I_1 - I_2 = \begin{cases} \frac{1}{2} - \frac{1}{2} & = \quad 0, \qquad t < -a\,, \\ \frac{1}{2} - \left(-\frac{1}{2}\right) & = \quad 1, \qquad -a < t < a\,, \\ -\frac{1}{2} - \left(-\frac{1}{2}\right) & = \quad 0, \qquad t > a\,. \end{cases}$$

For $|t| = a$, either $I_1 = 0$ or $I_2 = 0$ by symmetry. In either case we find $I = \frac{1}{2}$.

4. We wish to evaluate the integral

$$\Psi(\mathbf{k}) = (2\pi)^{-3/2}\,(32\pi a_0^5)^{-1/2} \int d^3\mathbf{x}\; z e^{-r/2a_0}\; e^{-i\mathbf{k}\cdot\mathbf{x}}\,,$$

where $\mathbf{x} = (x, y, z)$, $r = |\mathbf{x}|$, and we will omit the factor $(2\pi)^{-3/2}\,(32\pi a_0^5)^{-1/2}$ for clarity during the calculation .

The presence of r in the exponent suggests the use of spherical polar coordinates (r, θ, ϕ). In order to simplify the $e^{-i\mathbf{k}\cdot\mathbf{x}}$ term, it is convenient to choose the polar axis ($\theta = 0$) to lie not along the z axis of the original coordinate system as is conventional, but instead parallel to the vector $\mathbf{k}$, so $\mathbf{k}\cdot\mathbf{x} = kr\cos\theta$, where $k = |\mathbf{k}|$. The price of this simplification is that the expression for z becomes more complicated. Choosing the origin of ϕ so that the original z-axis lies in the $\phi = 0$ plane, it follows from elementary geometry that $\hat{\mathbf{z}} = (\sin\theta_k, 0, \cos\theta_k)$, where θ_k is the angle between $\hat{\mathbf{z}}$ and $\mathbf{k}$, so $z = \mathbf{x}\cdot\hat{\mathbf{z}} = r\sin\theta\cos\phi\sin\theta_k + r\cos\theta\cos\theta_k$. Hence we have

$$\Psi(\mathbf{k}) = \int_0^\infty r^2\,dr \int_{-\pi}^{\pi} \sin\theta\,d\theta \int_0^{2\pi} d\phi\; (r\sin\theta\cos\phi\sin\theta_k + r\cos\theta\cos\theta_k)\; e^{-r/2a_0 - ikr\cos\theta}\,.$$

Doing the ϕ integral, we see that the first term in parentheses integrates to zero, while the second gives 2π. Writing $\mu = \cos\theta$, the transform becomes

$$\Psi(\mathbf{k}) = 2\pi\cos\theta_k \int_0^\infty dr\; r^3\,e^{-r/2a_0} \int_{-1}^{1} d\mu\;\mu\,e^{-ikr\mu}\,.$$

The μ integral can be done by parts to give

$$\frac{2i}{kr}\left[\cos kr - \frac{\sin kr}{kr}\right],$$

so

$$\begin{aligned}\Psi(\mathbf{k}) \quad &= \quad \frac{4\pi i\cos\theta_k}{k}\left[\int_0^\infty dr\; r^2\cos kr\,e^{-r/2a_0}\right. \\ &\qquad\qquad \left. - \frac{1}{k}\int_0^\infty dr\; r\sin kr\,e^{-r/2a_0}\right].\end{aligned}$$

We now write $\cos kr = \frac{1}{2}(e^{ikr} + e^{-ikr})$ and $\sin kr = \frac{1}{2i}(e^{ikr} - e^{-ikr})$, and define the integrals

$$I_{n\pm} = \int_0^\infty dr\; r^n\, e^{\pm ikr - r/2a_0},$$

so our expression becomes

$$\Psi(\mathbf{k}) = \frac{2\pi i \cos\theta_k}{k}\left[I_{2+} + I_{2-} + \frac{i}{k}(I_{1+} - I_{1-})\right]$$

It is easily shown (by integration by parts and recursion) that

$$I_{n\pm} = \frac{n!}{(\frac{1}{2a_0} \mp ik)^{n+1}}\,,$$

so

$$\begin{aligned} I_{1+} - I_{1-} &= \frac{32\, ika_0^3}{(1+4k^2a_0^2)^2} \\ I_{2+} + I_{2-} &= \frac{32\, a_0^3(1 - 12\, k^2a_0^2)}{(1+4k^2a_0^2)^3}\,, \end{aligned}$$

and hence (still without the leading overall factor)

$$\Psi(\mathbf{k}) = -\frac{1024\, \pi\, i\, a_0^5\, k \cos\theta_k}{(1+4k^2a_0^2)^3}\,.$$

Alternatively, we can do the r integral first in the double integral above for $\Psi(\mathrm{k})$ to obtain

$$\begin{aligned} \Psi(\mathbf{k}) &= 2\pi\cos\theta_k \int_{-1}^{1} \frac{6\mu\; d\mu}{(\frac{1}{2a_0} + ik\mu)^4} \\ &= 192\,\pi a_0^4 \cos\theta_k \int_{-1}^{1} \frac{\mu\; d\mu}{(1 + 2ia_0k\mu)^4} \\ &= 192\,\pi a_0^4 \cos\theta_k \left(\frac{-16\, ika_0}{3(1+4k^2a_0^2)^3}\right) \\ &= -\frac{1024\,\pi\, i\, a_0^5\, k\cos\theta_k}{(1+4k^2a_0^2)^3}\,. \end{aligned}$$