

# PHYS 501: Mathematical Physics I

## Fall 2018, Homework #3

(Due October 23, 2018)

1. (a) Find the series solution of the equation

$$(1 - x^2)y'' - xy' + n^2y = 0$$

that is regular at  $x = 0$ . Under what circumstances (for what values of  $n$ ) does the series converge for *all*  $x$ ?

- (b) Find two linearly independent solutions of the equation

$$4x^2y'' + (1 - p^2)y = 0.$$

- (c) Given that one solution of the differential equation

$$y'' - 2xy' = 0$$

is  $y(x) = 1$ , use the Wronskian development to find a second, linearly independent solution. Describe its behavior near  $x = 0$ .

2. A function  $f(x)$  is periodic with period  $2\pi$ , and can be written as a polynomial  $P(x)$  for  $-\pi < x < a$  and as a polynomial  $Q(x)$  for  $a < x < \pi$ . Show that the Fourier coefficients  $c_n$  of  $f$  go to zero at least as fast as  $1/n^2$  as  $n \rightarrow \infty$  if  $P(a) = Q(a)$  and  $P(-\pi) = Q(\pi)$  (i.e.  $f$  is continuous), but only as  $1/n$  otherwise.

3. (a) Find the Fourier series  $\sum_{n=1}^{\infty} b_n \sin(n\pi x)$ , for  $-1 < x < 1$ , for the sawtooth function

$$f(x) = \begin{cases} -1 - x & (-1 < x < 0) \\ 1 - x & (0 < x < 1). \end{cases}$$

(b) Plot the partial sums  $S_N(x) = \sum_{n=1}^N b_n \sin(n\pi x)$  of the series for  $0 \leq x \leq 1$ , in steps of  $\delta x = 0.0005$ , and  $N = 1, 5, 10, 20, 50, 100$ , and  $500$ . What is the maximum overshoot of the Fourier series relative to the original function in the  $N = 500$  case, and at what value of  $x$  does it occur?

4. (a) Show explicitly from the series solutions that

$$\begin{aligned} J_{1/2}(x) &= A x^{-1/2} \sin x \\ J_{-1/2}(x) &= B x^{-1/2} \cos x. \end{aligned}$$

Hence, taking  $A = B = 1$  and using the recurrence relation  $J_{m+1}(x) = (2m/x)J_m(x) - J_{m-1}(x)$ , write down expressions for  $J_{3/2}(x)$  and  $J_{5/2}(x)$ .

- (b) A function  $f(x)$  is expressed as a Bessel series

$$f(x) = \sum_{n=1}^{\infty} a_n J_m(\alpha_{mn}x),$$

where  $\alpha_{mn}$  is the  $n$ -th root of  $J_m$ . Prove the Parseval relation

$$\int_0^1 [f(x)]^2 x dx = \frac{1}{2} \sum_{n=1}^{\infty} a_n^2 [J_{m+1}(\alpha_{mn})]^2.$$

5. The curved surface of a long cylinder of radius  $b$  is kept at a constant temperature  $T = 0$ . Initially the cylinder is at a uniform temperature  $T_0 > 0$ . Derive an expression for the temperature at the center of the cylinder at any time  $t > 0$ , and write down a simplified solution (not  $T = 0$ !) valid in the limit  $t \gg b^2/\kappa$ , where  $\kappa$  is the heat diffusion coefficient of the cylinder.