PHYS 501: Mathematical Physics I

Fall 2018

Solutions to Homework #3

1. (a) We seek a series solution of the ODE $(1-x^2)y'' - xy' + n^2 y = 0$ in the form

$$y(x) \;=\; x^k \sum_{m=0}^{\infty} c_m\, x^m \;=\; \sum_{m=0}^{\infty} c_m\, x^{m+k}\,,$$

where $c_0 \neq 0$. Substituting the sum into the differential equation yields

$$\begin{aligned}
&\sum_{m=0}^{\infty} (m+k)(m+k-1)\, c_m\, x^{m+k-2} \\
&\quad - \sum_{m=0}^{\infty} (m+k)(m+k-1)\, c_m\, x^{m+k} \\
&\quad\quad - \sum_{m=0}^{\infty} (m+k)\, c_m\, x^{m+k} \\
&\quad\quad\quad + \sum_{m=0}^{\infty} n^2\, c_m\, x^{m+k} \quad = \quad 0\,,
\end{aligned}$$

or, collecting terms

$$\begin{aligned}
&\sum_{m=-2}^{\infty} \quad (m+k+2)(m+k+1)\, c_{m+2}\, x^{m+k} \\
&- \sum_{m=0}^{\infty} \quad \Big\{ (m+k)(m+k-1)\, c_m\, x^{m+k} \\
&\qquad\qquad + (m+k)\, c_m\, x^{m+k} \\
&\qquad\qquad\qquad - n^2\, c_m\, x^{m+k} \Big\} \;=\; 0\,.
\end{aligned}$$

The leading term (x^{k-2}, from $m=-2$ in the first sum) gives the indicial equation

$$k(k-1) = 0\,,$$

so $k=0$ or 1. For $k=0$ the next term $[(k+1)k\, c_1]$ is automatically zero, so there is no constraint on c_1. For $k=1$, we must have $c_1 = 0$. The remaining terms imply

$$(m+k+2)(m+k+1)\, c_{m+2} \;=\; [(m+k)^2 - n^2]\, c_m\,,$$

connecting even to even and odd to odd terms. Obviously, the odd terms in the $k=0$ case, starting with $c_1 x$, give the same sequence as the even terms in the $k=1$ case, starting with $c_0 x$. Accordingly, we can consider the odd and even series separately. Both are regular at $x=0$.

Since

$$c_{m+2} \;=\; \frac{(m+k)^2 - n^2}{(m+k+2)(m+k+1)}\, c_m \,,$$

we see that $\lim_{n\to\infty} c_{m+2}/c_m = 1$, and the ratio test shows that each series has radius of convergence 1; in fact, both converge for $|x| = 1$ (see Arfken & Weber, §5.2). Both series diverge for $|x| > 1$ unless n is an integer, in which case the series terminate at $m = n - k$. (The solution in this case is the Chebyshev polynomial T_n.)

(b) We again seek a series solution of the form

$$y(x) \;=\; \sum_{m=0}^{\infty} c_m\, x^{m+k} \,.$$

Because the differential equation $4x^2y'' + (1-p^2)y = 0$ is homogeneous, substituting this series into the equation implies that

$$[4(m+k)(m+k-1) + (1-p^2)]c_m = 0$$

for all m. Since $c_0 \neq 0$, we obtain

$$4k(k-1) + 1 - p^2 = 0\,,$$

so

$$k = \tfrac{1}{2}(1 \pm p)\,.$$

For $m > 0$, we find

$$4m(m \pm p)c_m = 0\,,$$

so $c_m = 0$ unless $p = \mp m$, in which case $m + k = \frac{1}{2}(1 \mp p)$, that is, the non-vanishing term is just the other power-law solution. Thus the two solutions are

$$y(x) \;=\; x^{\frac{1}{2}(1\pm p)}\,,$$

and these are easily shown to be independent by computing their Wronskian.

(c) The first solution of

$$y'' - 2xy' = 0$$

is $y_1(x) = 1$. The Wronskian development gives, for the second solution

$$y_2(x) = y_1(x) \int^x e^{-\int^{x_2} P(x_1)\, dx_1}\, dx_2\,,$$

where $P(x) = -2x$ here. Thus

$$y_2(x) = \int^x e^{x_2^2}\, dx_2 = C + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!(2n+1)}\,,$$

where C is a constant. Near $x = 0$, $y_2 \sim C + x$.

2. We can write $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$, where

$$2\pi c_n = \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \int_{-\pi}^{a} P(x) e^{-inx} dx + \int_{a}^{\pi} Q(x) e^{-inx} dx \, .$$

(It is convenient to work with the exponential form of the series. The result applies equally well to the trigonometric form.) Assuming that P' and Q' exist (which is certainly the case if P and Q are polynomials), integration by parts gives

$$\begin{aligned} 2\pi c_n &= \left[\frac{P(x)}{-in} e^{-inx} \right]_{-\pi}^{a} + \int_{-\pi}^{a} \frac{P'(x)}{in} e^{-inx} dx \\ &\quad + \left[\frac{Q(x)}{-in} e^{-inx} \right]_{a}^{\pi} + \int_{a}^{\pi} \frac{Q'(x)}{in} e^{-inx} dx \\ &= \frac{e^{-ina}}{in} [Q(a) - P(a)] + \frac{1}{in} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx \, , \end{aligned}$$

where we have used the fact that $P(-\pi) = Q(\pi)$, by periodicity. If f is discontinuous at $x = a$, then the first term is nonzero and $c_n \sim 1/n$. Otherwise, the first term is zero, and similar arguments applied to f' show that c_n goes to zero at least as fast as $1/n^2$.

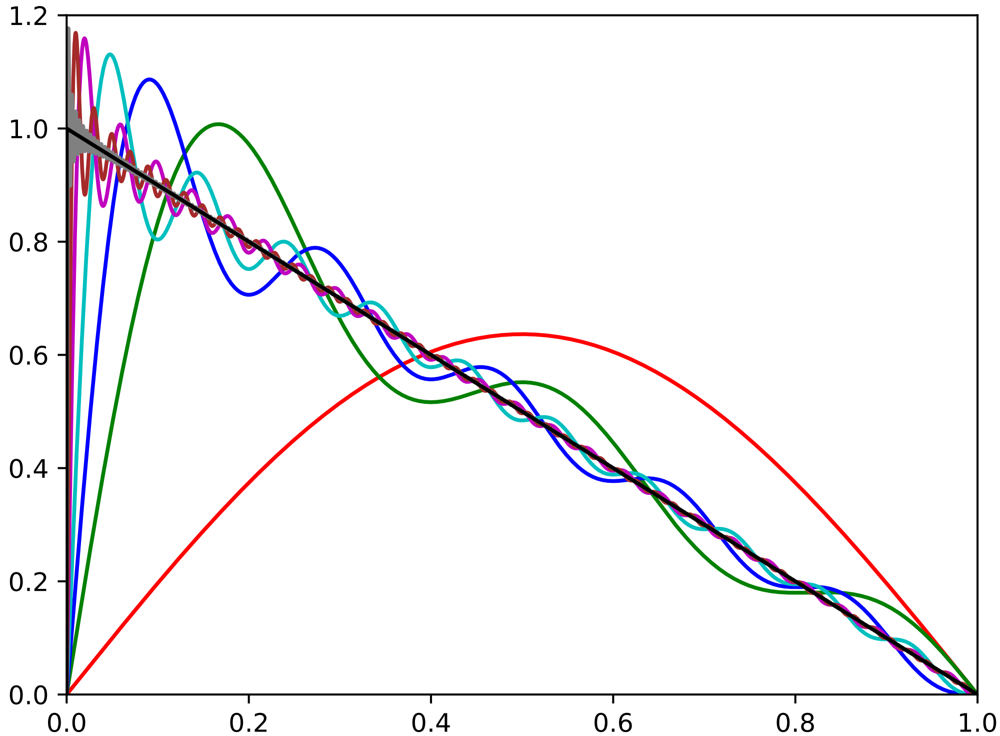
3. The function is odd, so we expect a Fourier sine series

$$f(x) = \sum_{n=0}^{\infty} a_n \sin n\pi x,$$

where

$$a_n = 2 \int_0^1 (1 - x) \sin n\pi x \, dx = \frac{2}{n\pi}.$$

The partial Fourier sums for the requested N values are shown in the figure below. Note the almost constant overshoot at $x \approx 1/N$, extending down even as far as $N = 1$.



4. (a) Bessel's equation is

$$x^2y + xy + (x^2 - m^2)y = 0\,.$$

Seeking a series solution of the form $y(x) = x^\alpha \sum_{n=0}^{\infty} c_n x^n$ and substituting into the equation we find, as usual

$$\begin{aligned} \alpha &= \pm m\,, \\ c_1 &= 0\,, \\ c_n &= \frac{-c_{n-2}}{(n+\alpha)^2 - m^2}\,. \end{aligned}$$

(i) For $m = \frac{1}{2}, \alpha = \frac{1}{2}$, we have $(n+\alpha)^2 - m^2 = (n+1)n$, so

$$\begin{aligned} c_{2k} &= \frac{(-1)^k c_0}{(2k+1)!}\,, \\ J_{\frac{1}{2}}(x) &= c_0 x^{\frac{1}{2}}(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\cdots) \\ &= c_0 x^{-\frac{1}{2}} \sin x\,. \end{aligned}$$

(ii) For $m = \frac{1}{2}, \alpha = -\frac{1}{2}$, we have $(n+\alpha)^2 - m^2 = n(n-1)$, so

$$\begin{aligned} c_{2k} &= \frac{(-1)^k c_0}{2k!}, \\ J_{\frac{1}{2}}(x) &= c_0 x^{-\frac{1}{2}}(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots) \\ &= c_0 x^{-\frac{1}{2}} \cos x\,. \end{aligned}$$

Using the recurrence relation $J_{m+1}(x) = (2m/x)J_m(x) - J_{m-1}(x)$ (and setting each $c_0 = 1$ for simplicity), we find

$$\begin{aligned} J_{\frac{3}{2}}(x) &= x^{-1} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x) &= x^{-\frac{3}{2}}(\sin x - x\cos x) \\ J_{\frac{5}{2}}(x) &= 3x^{-1} J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x) &= x^{-\frac{5}{2}}(3\sin x - 3x\cos x - x^2 \sin x)\,. \end{aligned}$$

(b) Writing $f(x) = \sum_{n=1}^{\infty} a_n J_m(\alpha_{mn} x)$, we have

$$\int_0^1 [f(x)]^2 x\,dx = \int_0^1 \left[\sum_{n=1}^{\infty} a_n J_m(\alpha_{mn}x)\right]\left[\sum_{k=1}^{\infty} a_k J_m(\alpha_{mk}x)\right] x\,dx\,.$$

Because of the orthogonality condition

$$\int_0^1 J_m(\alpha_{mn}x) J_m(\alpha_{mk}x) x\,dx = \tfrac{1}{2}[J_{m+1}(\alpha_{mn})]^2$$

(see H&R §9.5.3) only the terms with $k = n$ survive, so

$$\int_0^1 [f(x)]^2\, x\,dx = \frac{1}{2}\sum_{n=1}^{\infty} a_n^2\,[J_{m+1}(\alpha_{mn})]^2\,.$$

5. The temperature satisfies the diffusion equation

$$\frac{\partial T}{\partial t} = \kappa\,\nabla^2 T.$$

Separating out the time dependence $T(\mathbf{x}, t) = \chi(\mathbf{x}) e^{-\kappa k^2 t}$, we have

$$\nabla^2 \chi + k^2 \chi = 0,$$

with χ regular as $r = |\mathbf{x}| \to 0$ and $\chi = 0$ at $r = b$. The general solution is a sum of terms of the form

$$\chi(r, \phi) = J_m(kr)\, e^{im\phi},$$

where we have assumed that the solution is independent of z. The axisymmetric initial and boundary conditions imply that only the $m = 0$ term contributes, and the boundary condition at $r = b$ implies $J_0(kb) = 0$, so $k = k_n = \alpha_{0n}/b$, where α_{mn} is the n-th root of J_m. Thus the solution is

$$T(r, t) = \sum_n a_n\, J_0(k_n r)\, e^{-\kappa k_n^2 t}.$$

We determine the a_n by satisfying the initial conditions:

$$u(r,0) = T_0 = \sum_n a_n \, J_0\left(\frac{\alpha_{0n} r}{b}\right).$$

Inverting this Bessel series gives

$$a_n = \frac{2T_0}{b^2 J_1^2(\alpha_{0n})} \int_0^b J_0\left(\frac{\alpha_{0n} r}{b}\right) r dr.$$

We can evaluate the integral using the recurrence relation $xJ_0(x) = [xJ_1(x)]'$, to find

$$\begin{aligned}\int_0^b J_0\left(\frac{\alpha_{0n} r}{b}\right) r dr &= \frac{b^2}{\alpha_{0n}^2} \int_0^{\alpha_{0n}} sJ_0(s)\, ds \\ &= \frac{b^2}{\alpha_{0n}^2} \int_0^{\alpha_{0n}} [sJ_1(s)]'\, ds \\ &= \frac{b^2}{\alpha_{0n}} J_1(\alpha_{0n}),\end{aligned}$$

resulting in

$$a_n = \frac{2T_0}{\alpha_{0n} J_1(\alpha_{0n})}.$$

The central temperature is

$$\begin{aligned}T(0,t) &= \sum_n a_n \, e^{-\kappa k_n^2 t} \\ &\approx a_1 \, e^{-\kappa k_1^2 t} = \frac{2T_0}{\alpha_{01} J_1(\alpha_{01})} e^{-\kappa \alpha_{01}^2 t/b^2},\end{aligned}$$

where the leading term dominates the sum if

$$\kappa t(\alpha_{02}^2 - \alpha_{01}^2)/b^2 \gg 1,$$

or (since $\alpha_{01} = 2.40$, $\alpha_{02} = 5.52$)

$$t \gg \frac{b^2}{24.7\kappa}.$$