The Two-body Problem

The two-body problem—determining the motion of two bodies orbiting one another under their mutual gravitational attraction—is perhaps the best-known problem in gravitational dynamics. Here we show how it can be reduced to an equivalent one-body problem and then solved as a special case of motion in a potential field.

The equations of motion for two bodies of masses m_1 and m_2, positions $\mathbf{x}_1$ and $\mathbf{x}_2$, and velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ moving under gravity are

$$
\begin{aligned}
m_1\mathbf{a}_1 &= \frac{Gm_1m_2}{r_{12}^3}\left(\mathbf{x}_2-\mathbf{x}_1\right) &\quad (1)\\
m_2\mathbf{a}_2 &= \frac{Gm_1m_2}{r_{12}^3}\left(\mathbf{x}_1-\mathbf{x}_2\right), &\quad (2)
\end{aligned}
$$

where $\mathbf{a} = \ddot{\mathbf{x}}$ and $r_{12} = |\mathbf{x}_1 - \mathbf{x}_2|$. Adding these two equations, we see immediately that $m_1\mathbf{a}_1 + m_2\mathbf{a}_2 = 0$, so (integrating once) $m_1\mathbf{v}_1 + m_2\mathbf{v}_2 =$ constant and the center of mass of the bodies moves at constant velocity. Dividing Equation (1) by m_1, Equation (2) by m_2, and subtracting, we obtain the equation of motion for the relative separation $\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1$:

$$\ddot{\mathbf{r}} = -\frac{GM\mathbf{r}}{r_{12}^3} \tag{3}$$

where $M = m_1 + m_2$. Thus the relative motion in the two-body problem is identical to motion of a test particle in the potential field of a mass M.

Motion in a central potential [$\phi(r)$ say] is planar, since angular momentum is conserved, and in polar coordinates (r, θ) in that plane, the equations of motion are

$$
\begin{aligned}
\ddot{r} - r\dot{\theta}^2 &= F(r) &\quad (4)\\
r^2\dot{\theta} &= L \;=\; \text{constant}, &\quad (5)
\end{aligned}
$$

where $F(r) = -\phi'(r)$ and the second expression simply states conservation of angular momentum. We can use the angular momentum integral (5) to simplify the radial equation (4) by eliminating the transverse motion $\dot{\theta}$:

$$\ddot{r} \;=\; \frac{L^2}{r^3} + F(r) \;=\; -\frac{d\phi_{\text{eff}}}{dr}, \tag{6}$$

where the *effective potential* $\phi_{\text{eff}} = \phi + \frac{1}{2}L^2/r^2$ combines the external and the centrifugal forces into a single quantity.

We can also use Equation (5) to eliminate time as a variable in Equation (4) by writing

$$\frac{d}{dt} \;=\; \frac{L}{r^2}\frac{d}{d\theta} \tag{7}$$

so the radial equation becomes

$$\frac{L^2}{r^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) - \frac{L^2}{r^3} \;=\; F(r). \tag{8}$$

We now make the key substitution $u = 1/r$, so

$$\frac{dr}{d\theta} \;=\; -\frac{1}{u^2}\frac{du}{d\theta} \;=\; -r^2\frac{du}{d\theta} \tag{9}$$

and the radial equation (8) becomes

$$\frac{d^2u}{d\theta^2} + u \;=\; -\,\frac{F(1/u)}{L^2u^2}. \tag{10}$$

The first integral of this equation, obtained by multiplying by $du/d\theta$ and integrating wth respect to θ (and noting that $d\phi/dr = u^2 d\phi/du$) is

$$\tfrac{1}{2}\left(\frac{du}{d\theta}\right)^2 + \tfrac{1}{2}u^2 + \frac{\phi}{L^2} \;=\; \text{constant}. \tag{11}$$

This is often called the *radial energy equation.* Since

$$\frac{du}{d\theta} \;=\; \frac{1}{Lu^2}\frac{du}{dt} \;=\; -\,\frac{1}{L}\frac{dr}{dt} \;=\; -\,\frac{v_r}{L}, \tag{12}$$

where v_r the radial velocity, and since the transverse velocity is $v_\theta = r\dot{\theta} = L/r = Lu$, it should be clear that the constant on the right side of Equation (11) is just E/L^2, where

$$E \;=\; \tfrac{1}{2}v_r^2 + \tfrac{1}{2}v_\theta^2 + \phi(r) \tag{13}$$

is the energy per unit mass.

The equations become particularly simple for the so-called *Kepler problem* describing motion under gravity, since

$$F(r) \;=\; -\,\frac{GM}{r^2} \;=\; -\,GMu^2 \tag{14}$$

and Equation (8) becomes

$$\frac{d^2u}{d\theta^2} + u \;=\; \frac{GM}{L^2}. \tag{15}$$

The solution to Equation (15) is

$$u \;=\; C\cos(\theta - \theta_0) + \frac{GM}{L^2}, \tag{16}$$

where C and θ_0 are constants. Writing

$$e \;=\; \frac{CL^2}{GM} \tag{17}$$

$$a \;=\; \frac{L^2}{GM(1-e^2)}, \tag{18}$$

Equation (16) becomes

$$r\left[1 + e\cos(\theta - \theta_0)\right] \;=\; a(1-e^2), \tag{19}$$

which is the standard equation of an ellipse of semi-major axis a and eccentricity e, with the mass M at one focus and the long axis oriented along the direction $\theta = \theta_0$. For $e < 1$, r is bounded and the orbit is a closed ellipse. For $e \geq 1$ it is possible for $r \to \infty$ and the orbit is open—a hyperbola ($e > 1$) or a parabola ($e = 1$). By substituting Equation (16) into the radial energy equation (11) and using equations (17) and (18), it is easily verified that the total energy (per unit mass) of the motion is

$$E \;=\; -\,\frac{GM}{2a}. \tag{20}$$

The shape of the ellipse is completely specified by the geometric parameters a and e, or, equivalently, by the dynamical parameters E and L. Note that a hyperbolic solution with $e > 1$ corresponds to $a < 0$ and $E > 0$, so the dividing line between bound and unbound motion is $E = 0$. We can use Equation (20) to eliminate C from Equation (17) to find

$$e^2 \;=\; 1 + \frac{2EL^2}{G^2M^2}. \tag{21}$$

It is not possible to write down a closed-form analytic expression for r as a function of time t, but we can derive a parametric solution to the problem, as follows. We assume $E < 0$. A similar derivation holds for the unbound case. Writing

$$E \;=\; \tfrac{1}{2}v_r^2 + \tfrac{1}{2}v_\theta^2 - \frac{GM}{r} \;=\; \tfrac{1}{2}v_r^2 + \tfrac{1}{2}\frac{L^2}{r^2} - \frac{GM}{r}, \tag{22}$$

we can rearrange to find

$$v_r^2 \;=\; 2\left(E + \frac{GM}{r}\right) - \frac{L^2}{r^2} \tag{23}$$

$$\;=\; -\frac{2E}{r^2}\left(-r^2 - \frac{GM}{2E}r + \frac{L^2}{2E}\right). \tag{24}$$

But the solutions to this quadratic in r are known, since $v_r = 0$ at the two turning points of the orbit, corresponding to pericenter, $\theta = \theta_0, r = r_p = a(1-e)$, and apocenter, $\theta = \theta_0 \pm \pi, r = r_a = a(1+e)$. The quadratic must therefore factorize as $(r_a - r)(r - r_p)$, and we have

$$v_r^2 \;=\; -\frac{2E}{r^2}(r_a - r)(r - r_p). \tag{25}$$

Since $v_r = dr/dt$, we can say

$$t \;=\; \int \frac{dr}{v_r} \tag{26}$$

$$\;=\; \frac{1}{\sqrt{-2E}} \int \frac{r\,dr}{\sqrt{(r_a - r)(r - r_p)}}. \tag{27}$$

We can do the integral by using the substitution $r = a(1 - e\cos\eta)$. Then $dr = ae\sin\eta\,d\eta$ and $r_a - r = ae(1 + \cos\eta)$, $r - rp = ae(1 - \cos\eta)$ so the denominator is $ae\sin\eta$. The result is

$$t \;=\; \frac{a}{\sqrt{-2E}} \int (1 - e\cos\eta)\,d\eta \tag{28}$$

$$\;=\; \frac{a}{\sqrt{-2E}}(\eta - e\sin\eta). \tag{29}$$

Note that the definition of η means that $v_r < 0$ for $\eta < 0$ and $v_r > 0$ for $\eta > 0$, so this result is correct over the entire orbit.

Writing $E = -GM/2a$, the leading coefficient is $\sqrt{a^3/GM} \equiv \Omega^{-1}$, where Ω is the *mean motion*. Thus we have a complete parametric solution

$$\Omega t \;=\; \eta - e\sin\eta \tag{30}$$

$$r \;=\; a(1 - e\cos\eta). \tag{31}$$

Equation (30) is called *Kepler's equation.* It must be solved numerically for η given t. The parameter η is called the *eccentric anomaly.* Once r is known from Equation (31), θ can be determined from Equation (19). Finally, we note that the period of the motion is

$$T = \frac{2\pi}{\Omega} = 2\pi \left(\frac{a^3}{GM} \right)^{1/2}, \tag{32}$$

which is Kepler's Third Law.