Dynamical Friction and the Sinking Satellite Problem

In class we discussed how a massive body moving through a sea of much lighter particles tends to create a "wake" behind it, and the gravity of this wake always acts to decelerate its motion. The simplified argument presented in the text assumes small angle scattering and then integrates over all impact parameters from $b_{min} = b_{90}$ to b_{max}, the scale of the system. The result, for a body of mass M moving with speed V,

$$\frac{d\mathbf{V}}{dt} = -\frac{4\pi G^2 M \rho \ln \Lambda}{V^3} \mathbf{V}, \tag{1}$$

where $\Lambda = b_{max}/b_{min}$, as usual. This deceleration is called *dynamical friction.* A few notes on this equation are in order:

1. The direction of the acceleration is always *opposite* to the velocity of the massive body.

2. The effect depends only on the *density* ρ of the background, not on the individual particle masses. That means that the effect is the same whether the light particles are stars, black holes, brown dwarfs, or dark matter particles. The gravitational dynamics is the same in all cases. In practice, as we have seen, for a globular cluster or satellite galaxy moving through the Galactic halo, dark matter dominates the density field.

3. The acceleration drops off rapidly (as V^{-2}) as V increases.

4. This equation appears to predict that the acceleration becomes infinite as $V \to 0$. This is not in fact the case, and stems from the fact that we haven't taken the motion of the background particles properly into account. Binney and Tremaine (2008) do a more careful job of deriving Eq. 1, taking into account a distribution of particle velocities $f(v)$, and find

$$\frac{d\mathbf{V}}{dt} = -\frac{4\pi G^2 M \rho \ln \Lambda}{V^3} \mathbf{V} \int_0^V 4\pi v^2 f(v) dv. \tag{2}$$

Note the detailed analysis reveals that only particles with $v < V$ contribute to the deceleration. If the upper limit on the v integral were extended to infinity, then we would have $\int_0^\infty 4\pi v^2 f(v) dv = 1$ and we would recover Eq. 1. Note that for small V, $f(v) \approx f(0)$ and $\int_0^V 4\pi v^2 f(v) dv \approx \frac{4\pi}{3} V^3 f(0)$, so the acceleration actually goes to zero proportional to V.

For a Maxwellian velocity distribution with velocity dispersion σ,

$$f(v) = \left(2\pi\sigma^2\right)^{3/2} e^{-v^2/2\sigma^2},$$

the integral can be evaluated:

$$\int_0^V 4\pi v^2 f(v) dv = \mathrm{erf}(X) - \frac{2X}{\sqrt{\pi}} e^{-X^2},$$

where $X = V/\sqrt{2}\sigma$ and the error function is defined as $\mathrm{erf}(y) \equiv \int_0^y e^{-z^2} dz$. It is common simply to evaluate this expression at $X = 1$, which is where the factor $\mathcal{F} = 0.428$ mentioned in Sparke and Gallagher problem 7.6 (p. 286) comes from.

As with the other dynamical processes we have considered, it is convenient to define a dynamical friction time scale t_{DF} by

$$\frac{d\mathbf{V}}{dt} = -\frac{\mathbf{V}}{t_{DF}},$$

so

$$t_{DF} = \frac{V^3}{1.7\pi G^2 M\rho \ln\Lambda},$$

where we have again used $\mathcal{F} = 0.428$. Note the similarity of this expression to the relaxation time for the background particles

$$t_r = \frac{v^3}{8\pi G^2 m\rho \ln\Lambda}.$$

However, since $M \gg m$ the dynamical friction time scale is orders of magnitude smaller than the relaxation time, and is less than the age of the Galaxy for many satellites.

We can now apply these ideas to the problem of a satellite galaxy (such as one of the Magellanic clouds) slowly sinking toward the center of our Galaxy due to dynamical friction. Assuming a dark-matter dominated halo with a flat rotation curve of circular velocity v_c, we can write, as usual,

$$v_c^2 = \frac{GM(r)}{r},$$

where $M(r)$ is the mass inside radius r:

$$M(r) = \frac{v_c^2 r}{G}.$$

For a spherically spherical Galaxy, we have

$$\frac{dM}{dr} = 4\pi r^2 \rho(r),$$

so the density profile of the halo is

$$\rho(r) = \frac{v_c^2}{4\pi G r^2}.$$

Let's idealize the calculation to the case of an initially circular orbit, and assume that the orbit decays slowly enough that we can regard the satellite as sinking through a series of circular orbits. At any instant, at distance r from the center, its angular momentum then is $L = rv_c$, and the rate of change of the angular momentum is equal to the applied torque, ra_{DF}, where a_{DF} is the acceleration due to dynamical friction,

$$\begin{aligned} a_{DF} &\approx -0.428\frac{4\pi G^2 M\rho(r)\ln\Lambda}{v_c^2} \\ &= -0.428\ln\Lambda\frac{GM}{r^2}. \end{aligned}$$

Since

$$\frac{dL}{dt} = v_c\frac{dr}{dt},$$

it follows that

$$v_c\frac{dr}{dt} = -0.428\ln\Lambda\frac{GM}{r}.$$

This is a differential equation for $r(t)$, the satellite's distance from the Galactic center, which is easily solved given the satellite's initial distance, $r(0) = R_0$, say. See Homework 5, problem 5.