PHYS 431/531: Galactic Astrophysics

Fall 2021, Homework #6

(Due November 20, 2021)

1. (a) Verify that the Kuzmin potential

$$\Phi_K(R,z) = -\frac{GM}{\sqrt{R^2 + (a+|z|)^2}}$$

has $\nabla^2\Phi = 0$ for $z \neq 0$, and so represents a surface density distribution $\Sigma(R)$ in the plane $z = 0$.

(b) Use Gauss's law to determine $\Sigma(R)$.

(c) What is the circular orbit speed for a particle moving in the plane of the disk?

2. Sparke & Gallagher, problem 3.25.

3. *[Undergraduates only]* Show that, if the distribution function in a spherically symmetric system is a function of energy only, then the velocity distribution is isotropic (i.e. $\langle v_x^2\rangle = \langle v_y^2\rangle = \langle v_z^2\rangle$).

4. *[Graduate students only]* A stellar system in which all particles are on radial orbits is described by the distribution function

$$f(\mathcal{E}, L) = \begin{cases} A\delta(L^2)(\mathcal{E} - \mathcal{E}_0)^{-1/2} & (\mathcal{E} > \mathcal{E}_0) \\ 0 & (\mathcal{E} \leq \mathcal{E}_0), \end{cases}$$

where $\mathcal{E} = \psi - \frac{1}{2}v^2$ is relative energy and $\mathcal{E}_0$ and A are constants.

(a) By writing $v^2 = v_r^2 + v_t^2$, where v_r and v_t are the radial and transverse velocities, and $L = rv_t$, prove that the volume element $d^3v = 2\pi v_t dv_t dv_r$ may be written

$$d^3v = \frac{\pi d\mathcal{E} dX}{r^2 v_r},$$

where $X = L^2$.

(b) Hence show that the density is

$$\rho(r) = \begin{cases} Br^{-2} & (r < r_0) \\ 0 & (r \geq r_0), \end{cases}$$

where B is a constant and the relative potential at r_0 satisfies $\psi(r_0) = \mathcal{E}_0$.

5. (a) Assuming that the Galaxy's rotation curve is flat in the neighborhood of the Sun, with velocity $V_0 = 220$ km/s at distance $R_0 = 8$ kpc from the Galactic Center, calculate the local epicyclic frequency κ. (b) If the Sun has v_x (radial) $= -10$ km/s and v_y (transverse) $= 5$ km/s, calculate the Sun's guiding radius R_g and radial orbital amplitude X. (c) Sketch the curves of Ω, $\Omega \pm \kappa/2$, and $\Omega \pm \kappa/4$ in a disk where $V(R)$ is constant everywhere, and show that the zone where two-armed spiral waves can persist is almost four times larger in area than that for four-armed spirals.