PHYS 231 Lecture Notes – Week 6

Reading from Maoz (2^{nd} edition):

- **Sections 4.1, 4.2**

Once again, a lot of the material presented in class this week is well covered in Maoz, and we simply reference the book, with additional comments and derivations as needed. References to slides on the web page are in the format "slidesx.y/nn," where x is week, y is lecture, and nn is slide number. Note: references to class slides and figures in the new Maoz edition have not yet been checked.

6.1 Leaving the Main Sequence

See Maoz §4.1.
As we have seen, stars stay on the main sequence for a time

$$t_{MS} \sim 10^{12} \text{ yr } \left(\frac{M}{M_\odot}\right)^{-3}.$$

This assumes $L \propto M^4$. For very massive stars, as we have seen, $L \propto M$, so t_{MS} is roughly constant, about 3–5 Myr. After this time, a star has burned a significant fraction of the hydrogen in its core into helium, and the composition of the central regions has changed radically. An inner core of pure helium begins to build up.

But there is a problem. The main sequence core temperature is around 10^7 K, which is far too low to overcome the coulomb barrier to fuse helium. Consequently, the core has no energy source to make up the energy which is still diffusing outward into the cooler overlying regions. In addition, as fusion turns hydrogen into helium, the mean molecular weight of the gas in the core increases, and the pressure decreases. As a result of these processes, the core contracts. As it contracts, it heats up (by the virial theorem), heating the overlying hydrogen layers and causing them to burn even more furiously. The star at this stage has a contracting, nonburning helium core surrounded by a shell of burning hydrogen (slides6.1/11). Its luminosity increases, its hydrogen envelope becomes convective, and its radius increases. It is on its way to becoming a red giant.

As a cluster of stars ages, we can imagine (slides2.2/05-09) stars being "peeled off" the main sequence from top to bottom as the helium core forms and grows in mass. They move up and to the right in the HR diagram. The *red giant branch* consists of stars of different masses at various stages of this process. One of the reasons we have such faith in the theory of stellar evolution is that the red giant branches predicted by theory agree so well with those we actually observe (slides6.1/15). The lines in those figures show *isochrones*—lines of constant cluster age, with different initial mass stars lying at different locations along the line. Carefully fitting isochrones to observed clusters provides a much more accurate estimate of cluster age and distance than simply trying to estimate which star corresponds to the turnoff mass.

Eventually the helium core becomes hot enough that helium fusion can occur—around 10^8 K. The density by then is around 10^7 kg m^{-3}. The new reaction is known as the triple-alpha process (helium nuclei are also known as alpha particles).

$$\begin{aligned} {}^4He + {}^4He &\longrightarrow {}^8Be \\ {}^8Be + {}^4He &\longrightarrow {}^{12}C + \gamma \end{aligned}$$

(see slides6.1/16). The rate-limiting step is the first reaction, in which two helium nuclei fuse to form beryllium, since it consumes energy, and so the beryllium nucleus quickly (after about 10^{-16}

s) breaks up into helium nuclei. However, if the temperature and density are high enough, the second reaction can occur during this brief time, forming carbon. This odd sequence of events has to occur because there are no other accessible stable nuclei between helium and carbon. The star by this time has moved up to the top of the red giant branch. Once helium fusion starts, it moves over to a “helium main sequence” of sorts, known as the horizontal branch.

On the horizontal branch, the star has a burning helium core surrounded by a hydrogen burning shell. Once significant carbon has formed, other elements can be created by the process of *alpha capture*:

$$\begin{aligned} {}^4He + {}^{12}C &\longrightarrow {}^{16}O + \gamma \\ {}^4He + {}^{16}O &\longrightarrow {}^{20}Ne + \gamma \\ {}^4He + {}^{20}Ne &\longrightarrow {}^{24}Mg + \gamma. \end{aligned}$$

These reactions can occur even though the core is still too cool to fuse carbon because they have a lower coulomb barrier.

As the helium core burns and carbon, oxygen, and heavier elements build up, history repeats itself. As just mentioned, carbon can’t fuse because the temperature is too low, so an inert and contracting carbon core builds up, surrounded by a helium burning shell and a hydrogen burning shell (see slides6.1/17). The carbon core contracts (for the same reason the helium core contracted when the star left the main sequence), heats up, accelerates the burning in the surrounding shells, and the star once again brightens and expands, heading back into the red giant region. Its track is called the *asymptotic giant branch*, and ultimately merges with the earlier red giant branch.

On the asymptotic giant branch, the star is convective and, this time, the convection reaches down into the core, dredging up heavy elements and mixing them into the envelope. This turns out to be an important source for new heavy elements in the universe, once the envelope is expelled into interstellar space.

At this point you might imagine that the same basic cycle will keep repeating—heavier elements are formed by fusion, build up in a contracting, nonburning core, which heats up and eventually ignites in a new round of burning. But instead, for some stars, new physics intervenes. As we will discuss in more detail next time, for most stars, the contracting core never heats up to the point where carbon can fuse. Instead, the carbon-oxygen core grows, the star gets larger and more luminous, and eventually becomes unstable. The envelope becomes detached from the star and expands into space, leaving the hot core behind. The expanding envelope, irradiated by ultraviolet radiation from the core, glows as a *planetary nebula* (bad terminology—nothing to do with planets!). Several examples are shown in slides6.1/20-23. The core is left behind as a hot, small, and low-luminosity *white dwarf*. Cluster color-magnitude diagrams (slides6.1/24) show a well-defined white-dwarf track.

A typical white dwarf has a mass of $\sim 0.5 - 1.0\,M_\odot$, a radius of $\sim 10^4$ km, and is composed predominantly of carbon and oxygen. More massive white dwarfs contain higher fractions of oxygen and neon.

6.2 Electron Degeneracy

See Maoz §4.2

Before the carbon core of a solar-mass star reaches a high enough density for carbon to fuse, the gas enters a new physical state, dominated by new processes.

Up to now, we have been dealing with ideal gases, loosely defined as gases in which the constituent particles can be treated as particles that interact elastically when they collide but which otherwise exert no interparticle forces. Think of tiny billiard balls. But if a gas is dense enough, that is no longer true. A basic prediction of quantum mechanics is that, on small scales, particles exhibit wave properties. The *de Broglie wavelength* of a particle with momentum p is just

$$\lambda = \frac{h}{p},$$

where h is Planck's constant. If the interparticle separation in the gas is comparable to or smaller than the de Broglie wavelength, then the gas is no longer ideal, but instead is governed by the rules of quantum mechanics.

Now consider electrons (mass m_e) in a fully ionized gas of mean particle mass m. In energy equipartition, clearly the lightest particles will have the largest momenta and hence the shortest de Broglie wavelengths, so we expect them to show quantum effects first as the density increases. If the gas has density ρ, the number density is $n = \rho/m$ and the mean interparticle distance is $\ell = n^{-1/3}$. If the temperature is T, the mean energy per particle is $E = \frac{3}{2}kT$ and the mean momentum per (nonrelativistic) electron is $p = \sqrt{2mE}$, so we take

$$\bar{\lambda} = \frac{h}{(3mkT)^{1/2}}.$$

Conventionally, we call a gas a quantum gas if

$$\ell \le \bar{\lambda}/2 \quad \Longrightarrow \quad \rho \ge \frac{8m}{h^3}\,(3m_e kT)^{3/2} \equiv \rho_q.$$

Plugging in the numbers for fully ionized pure hydrogen, we find

$$\rho_q = 3.4 \times 10^5\,\mathrm{kg/m^3} \left(\frac{m}{m_p}\right) \left(\frac{T}{10^7\,\mathrm{K}}\right)^{3/2}.$$

For comparison, in the current core of the Sun, with $T = 1.5 \times 10^7$ K, $\rho_q = 6.5 \times 10^5\,\mathrm{kg/m^3}$, more than 4 times the actual density there, so quantum effects are relatively unimportant. But increase the density by a factor of 10, and things would become very different. A gas in which quantum effects dominate is called *degenerate*.

People often think of degeneracy as meaning high density or low temperature, but note that even a very dense gas can still be non-degenerate if its temperature is high enough.

6.3 Fermi-Dirac Gases

Quantum mechanical particles are probabilistic things subject to the *uncertainty principle*, which states that, if the uncertainty in a particle's position is Δx and the uncertainty in its momentum is Δp, then (in 1 dimension)

$$\Delta x \Delta p \ge h.$$

In other words, if I somehow confine a particle so that I know its position to great accuracy, then the spread in its possible momentum becomes very large. And since pressure is an integral over momenta (see below), that means that the pressure associated with this momentum spread can be very large. In 3 dimensions, the uncertainty principle becomes

$$d^3x\, d^3p \ge h^3.$$

Earlier we encountered the Maxwell-Boltzmann distribution of particle speeds in a gas and the blackbody distribution of photon energies, both for systems in thermal equilibrium. In each case, the distribution was obtained via *statistical mechanics*, counting all possible ways in which a collection of particles can be in any given macroscopic energy state. We can do this with electrons (and protons and neutrons) too, but when we do this we have to include the additional fact that they are unlike other particles we have discussed so far in that they are subject to the *exclusion principle*, which basically says that no two electrons (or protons or neutrons) can exist in the same quantum state.

Following through the counting process subject to this condition, we find that the number of electrons dN in volume d^3x with momenta in the range d^3p (a density in the 6-dimensional *phase space* of positions and moments) is given by the *Fermi-Dirac distribution*:

$$dN \;=\; f(x,p)\, d^3x\, d^3p \;=\; \frac{2s+1}{e^{[E-\mu(T)]/kT+1}}\,\frac{d^3x\, d^3p}{h^3}.$$

Here, s is the spin of the particle ($s = \frac{1}{2}$ for all the particles considered here), and $\mu(T)$ is an energy called the *chemical potential*, which tends to a value called the *Fermi energy* E_f as $T \to 0$. The function f is called a *distribution function*—the 6-D equivalent of the more familiar density in 3-D space. Indeed, integrating f over all momenta gives us the number density of particles:

$$n \;=\; \int d^3p\, f(x,p).$$

Note that as $T \to \infty$ we find

$$dN \;\propto\; d^3x\, d^3p e^{-E/kT},$$

that is, the distribution function becomes a Maxwell-Boltzmann distribution (see week 3 notes, and note that $d^3p = 4\pi p^2 dp \;\propto\; 4\pi v^2 dv$), as we'd expect for an ideal gas in which quantum effects can be neglected.

Of greatest interest here is the distribution in the completely degenerate quantum limit, when the temperature $T \to 0$. As illustrated in slides6.2/10-11, for low temperatures the Fermi-Dirac distribution takes on a particularly simple form, with $f = 1$ for energies less than $E_F = \mu(0)$ and $f = 0$ for larger energies. In essence, the electrons try to pack themselves into the lowest allowed energy states, but since they limited by the exclusion principle, this means that they completely fill all states up to the Fermi energy, with no electrons above that energy. Assuming isotropic motion, so $d^3p = 4\pi p^2 dp$, and setting $s = \frac{1}{2}$, we have

$$dN \;=\; f(x,p)\, d^3x\, dp \;=\; \begin{cases} 8\pi p^2 dp\, d^3x/h^3, & p < p_F \\ 0, & p > p_F, \end{cases}$$

where the Fermi momentum p_F for nonrelativistic particles is given by

$$\frac{p_F^2}{2m_e} = E_F.$$

We can now write down an expression for the the electron number density:

$$n_e \;=\; \int d^3p\, f(x,p) \;=\; \int_0^{p_F} \frac{8\pi}{h^3} p^2\, dp \;=\; \frac{8\pi}{3h^3} p_F^3,$$

so the Fermi momentum p_F is determined by the number density:

$$p_F \;=\; \left(\frac{3h^3}{8\pi} n_e\right)^{1/3}.$$

Thus the electron density completely determines both the Fermi momentum and the Fermi energy for a fully degenerate gas:

$$E_F \;=\; \left(\frac{3}{8\pi}\right)^{2/3} \left(\frac{h^2}{m_e}\right) n_e^{2/3}.$$

6.4 Equation of State for Degenerate Matter

Next we need to calculate the pressure of the gas. Let's start by deriving a general expression for the pressure of a gas described by an isotropic distribution function. This applies to classical particles, relativistic particles, and quantum particles.

(Figure to come...)

First imagine a particle of momentum p striking a surface at angle θ and reflecting elastically from it, as illustrated in the above figure. The change in the particle's momentum, and the momentum imparted to the surface, is clearly $2p\cos\theta$.

Now imagine a beam of particles of cross-sectional area δA, number density n, speed v, and momentum p hitting the surface. The rate at which particles strike the surface is $nv\delta A$. The projected area on the surface over which they strike is $\delta A/\cos\theta$. Hence the total momentum imparted to the surface per unit area per unit time (i.e. the pressure due to these particles) is

$$\frac{d^2 p_{surf}}{dt\,dA} = (2p\cos\theta)\,(nv\cos\theta) = 2nvp\cos^2\theta.$$

If the particles are drawn from an isotropic distribution, the fraction of particles moving in this particular direction, into solid angle $d\Omega = \sin\theta d\theta d\phi$ is just $d\Omega/4\pi$. Integrating over all angles that intersect the surface, we get

$$\frac{d^2 p_{surf}}{dt\,dA} = \frac{1}{4\pi}\int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi\; 2nvp\cos^2\theta\,\sin\theta = \frac{1}{3}nvp.$$

Finally, including the distribution function, so $n = f(p)dp$, and integrating over momenta p, we obtain a general expression for the pressure (momentum per unit area per unit time):

$$P = \frac{1}{3}\int f(p)\,v(p)\,p\,dp.$$

This general expression reduces to the pressures we have already seen in the different circumstances discussed previously:

1. For an ideal nonrelativistic gas described by a Maxwell-Boltzmann distribution, with v(p) = p/m, we recover the thermal pressure $P_{th} = nkT$.
2. For radiation, with $v = c$ and $p = E/c$, integrating over the blackbody distribution gives $P_{rad} = \frac{1}{3}aT^4$.

For a degenerate quantum electron gas, the pressure is

$$\begin{aligned} P_e &= \frac{1}{3}\int_0^{p_F} f(p)\,v(p)\,p\,dp \\ &= \frac{1}{3}\int_0^{p_F} \frac{8\pi}{h^3}\frac{p^4}{m_e}\,dp \\ &= \frac{8\pi}{3h^3 m_e}\frac{p_F^5}{5} \\ &= \left(\frac{3}{8\pi}\right)^{2/3}\frac{h^2}{5m_e}\,n_e^{5/3}. \end{aligned}$$

We can relate n_e to the density ρ by writing the mass density is accounted for by a single element of atomic number Z and mass number A, so $n_e = Z(\rho/Am_p)$ and we have

$$P_e = \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20 m_e m_p^{5/2}} \left(\frac{Z}{A}\right)^{5/3} \rho^{5/3}.$$

This pressure has nothing to do with temperature, and is *independent* of the normal thermal pressure we normally associate with a gas.

Let's compare thermal and degenerate electron pressure for conditions in white dwarfs. A typical white dwarf has $\rho \sim 10^9$ kg/m^3 and $Z/A \sim 0.5$. The electron degeneracy pressure is then $P_e \sim 3 \times 10^{21}$ Pa. The thermal pressure due to nuclei (electrons would produce a similar number) for $T = 10^7$ K is $P_{th} = n_n kT \sim 2 \times 10^{19}$ Pa $\ll P_e$, so electron degeneracy pressure dominates.

6.5 Electron Degeneracy in Stellar Evolution

Electron degeneracy pressure plays multiple critical roles in determining the outcome of stellar evolution (see slides6.2/18). Under the right circumstances, electron degeneracy pressure can intervene to prevent every major evolutionary stage:

1. For masses less than about $0.08\,M_\odot$, electron degeneracy can support the hydrogen core of a would-be star before the temperature reaches the $\sim 10^7$ K needed to fuse hydrogen. Such low-mass failed stars are called *brown dwarfs*.

2. For masses less than about $0.25\,M_\odot$, electron degeneracy pressure will support the helium core of a red giant before helium fusion can start. The *helium white* dwarfs resulting from this process are currently not observed, because no $0.25\,M_\odot$ star has ever left the main sequence.

3. For masses less than about $8\,M_\odot$, electron degeneracy will support the carbon-oxygen core of a asymptotic giant branch star before further fusion can occur, resulting in a *carbon-oxygen white dwarf*.

4. For masses less than about $12\,M_\odot$, electron degeneracy will support the oxygen-neon core of a asymptotic giant branch star before further fusion can occur, resulting in a *neon-oxygen white dwarf*.

6.6 White Dwarf Mass-Radius Relation

See Maoz §4.2.3

Last time we saw how, under conditions of high density, electron degeneracy pressure can dominate "normal" thermal pressure. We can use the scaling relations to explore how this new equation of state affects the structure of these stars. Combining the relations

$$\begin{aligned} P &\sim \frac{M^2}{R^4} \\ \rho &\sim \frac{M}{R^3} \\ P &\sim \rho^{5/3} \sim \frac{M^{5/3}}{R^5} \end{aligned}$$

we find

$$R \sim M^{-1/3}.$$

Thus, more massive white dwarfs are smaller. Plugging in the neglected numbers, we find

$$
\begin{aligned}
R_{wd} &\sim \frac{h^2}{20 m_e m_p^{5/3} G} \left(\frac{Z}{A}\right)^{5/3} M^{-1/3} \\
R_{wd} &\approx 2.3 \times 10^4 \text{ km} \left(\frac{Z}{A}\right)^{5/3} \left(\frac{M}{M_\odot}\right)^{-1/3},
\end{aligned}
$$

where the coefficient in the final expression is the result of a more accurate calculation taking the structure of the dwarf properly into account.

6.7 The Chandrasekhar Mass

See Maoz §4.2.3
As the mass increases, the white dwarf shrinks and electron momenta increase, eventually becoming relativistic. If we repeat our earlier calculation of the pressure, but now setting $v = c$ instead of $v = p/m_e$, we obtain

$$
\begin{aligned}
P_e &= \frac{1}{3}\int_0^{p_F} f(p)\, v\, p\, dp \\
&= \frac{1}{3}\int_0^{p_F} \frac{8\pi}{h^3} c p^3\, dp \\
&= \frac{8\pi c}{3h^3}\frac{p_F^4}{4} \\
&= \left(\frac{3}{8\pi}\right)^{1/3} \frac{hc}{4} n_e^{4/3}.
\end{aligned}
$$

Again writing $n_e = Z(\rho/Am_p)$, we can write this in terms of the density:

$$
P_e = \left(\frac{3}{8\pi}\right)^{1/3} \frac{hc}{4 m_p^{4/3}} \left(\frac{Z}{A}\right)^{4/3} \rho^{4/3}.
$$

Note that m_e does not appear explicitly, which is reasonable as we have assumed $m_e c^2 \ll pc$.

This new equation of state presents a problem. If we set $P \sim \rho^{4/3}$ in the above structure relations, we find that the radius disappears from the mass-radius relation. We can be more systematic, and imagine an equation of state $P \sim \rho^{(4+\epsilon)/3}$, where $\epsilon = 1$ in the nonrelativistic regime and decreases to 1 as the star becomes relativistic. Using this expression for P, we find

$$
R \sim M^{(\epsilon-2)/3\epsilon},
$$

so $R \to 0$ as $\epsilon \to 0$. In other words, the star becomes unstable as it becomes relativistic. Electron degeneracy pressure can no longer support it against gravity. The maximum mass that the star can have before it becomes relativistic and unstable is known as the *Chandrasekhar mass*, M_{ch}, after the Nobel Prize winning Indian-American astrophysicist Subrahmanyan Chandrasekhar, who first derived this limit.

Using the virial theorem $PV = \frac{1}{3}E_{gr}$, writing $E_{gr} \sim GM^2/R$, $\rho = M/V$, $V = \frac{4}{3}\pi R^3$, and using the above relativistic expression for P, we find that R cancels and

$$
M_{ch} \sim 0.11 \left(\frac{Z}{A}\right)^2 \left(\frac{hc}{Gm_p^2}\right)^{3/2} m_p.
$$

A more precise calculation, taking the structure of the star properly into account, replaces the 0.22 by 0.21. Plugging in the numbers for $Z/A = 0.5$, appropriate for a C or CO white dwarf, we obtain

$$M_{ch} \;=\; 1.4\, M_\odot.$$

6.8 Degenerate Burning

Stars much more massive than the Sun start to fuse helium into carbon more or less uneventfully in their cores. However, in stars of masses between $0.25\, M_\odot$ and about $2\, M_\odot$, the core is significantly degenerate when helium fusion begins, meaning that electron degeneracy contributes a large fraction of the total pressure. Since electron degeneracy pressure is independent of temperature, which creates an unstable situation.

The onset of a new nuclear burning stage releases large amounts of energy into the core, raising the temperature. If the pressure were thermal, $P_{th} = (\rho/\bar{m})kT$, the rising temperature would cause the pressure to increase, causing the core to expand a little and stabilizing the burning (recall that the nuclear burning rates depend very sensitively on temperature). In a degenerate (or nearly degenerate) gas, the rising temperature is not offset by a large rise in pressure, so the compensating effect of thermal pressure is greatly reduced. As a result, the core does not expand and the temperature is not stabilized. Instead, the temperature and burning rates rise in a runaway process called the *helium flash*. The process lasts only a few seconds, but during that time nuclear emissivity may be billions of times higher than normal, and a large amount of energy is dumped into the core.

After the helium flash the temperature has risen to the point where degeneracy pressure is no longer important, the density is lower, and helium is burning non-degenerately at the center. The star jumps from the top of the giant branch to the horizontal branch, and the evolution continues as described earlier. The energy released by the flash is not seen from outside. Instead, it is used up in expanding the core and lifting the degeneracy.

Degenerate burning can also occur in white dwarfs, where it can have much more catastrophic consequences. When a white dwarf is pushed over the Chandrasekhar mass it starts to collapse, heats up, and undergoes degenerate carbon fusion (see the simulation on the course web page). The star may be pushed over the limit by accretion from a companion in a binary, or two white dwarfs may collide (for example, as their orbit decays by gravitational radiation emission, see HW5). Nuclear burning an a degenerate gas is unstable, and in this case the entire star explodes once the burning starts. As illustrated in slides6.2/28, the burning starts near the center, where the temperature is highest, and the hot burning region then rises to the surface, after which it engulfs the entire star. The result is called a Type Ia supernova. Its peak luminosity is more than a billion times that of the Sun (see slides6.2/24). Over its several-month lifetime, it will radiate away over 10^{44} J of energy—more than the Sun will generate over its entire 10 billion year lifetime. There is some debate about whether the process leaves a remnant behind. Most sources seem to think not, but a vocal minority claim that a neutron star (see below) may result. Slides6.2/25 shows an example of what remains after the supernova is over.

6.9 High-Mass Stars

See week 7 notes.