PHYS 231 Lecture Notes – Week 3

Reading from Maoz (2nd edition):

- **Chapter 2, Sec. 3.1, 3.2**

A lot of the material presented in class this week is well covered in Maoz, and we simply reference the book, with additional comments and derivations as needed. References to slides on the web page are in the format "slidesx.y/nn," where x is week, y is lecture, and nn is slide number.

3.1 Atoms and Radiation

Before moving on, let's look a little more on the interactions between matter and radiation, and the resulting emission and absorption spectra we see. This section goes considerably beyond the syllabus of the course, but it is intended to give some flavor of how emission and absorption processes at the atomic level relate to the large-scale radiation field in a star. We Consider a system in thermal equilibrium at temperature T. That means that all aspects of all atoms and ions — speeds, internal excitation state, and ionization state — are all determined by T. Now consider, as above, an idealized two-level atom, with levels 1 and 2, with $E_2 > E_1$, in that environment.

3.1.1 Emission

Let's first consider the atom's emission. An atom in the excited 2 state, left to its own devices, will eventually *spontaneously* emit a photon and drop to the lower 1 state, where $E_2 - E_1 = E_{21} = h\nu_0$, the frequency associated with the transition. The probability per unit time of such an occurrence is the *Einstein coefficient* A_{21} for the two levels. it is a quantum-mechanical property of the atom, and can in principle be calculated for any given system. The emission is isotropic, and we can easily write down an expression for the resulting emissivity (unit $\mathrm{W\,m^{-3}\,ster^{-1}}$):

$$j_\nu = h\nu\, A_{21}\, n_2\, \phi(\nu)/4\pi.$$

Here, $h\nu_0$ is the line energy (J), A_{21} is the transition probability per unit time ($\mathrm{s^{-1}}$), n_2 is the density of atoms in the upper state ($\mathrm{m^{-3}}$), and the final 4π means that j_ν is emission per steradian. The dimensionless factor $\phi(\nu)$ is the *line profile* — a delta-function spike for an idealized line, but in reality a sharply peaked, but broadened, function centered on ν_0, with $\int \phi(\nu) d\nu = 1$. This process gives rise to the emission lines we see.

Notice that observations of emission lines from different transitions in the same gas (for example, the hydrogen lines Hα and Hβ) can provide accurate information on the gas temperature. Let's compare emission lines from the $2-1$ and the $i-k$ transitions in the same atom. Temporarily ignoring the line profiles, the ratio of the line strengths is

$$\begin{aligned}
\frac{j_{21}}{j_{ik}} &= \frac{E_{21}}{E_{ik}}\frac{A_{21}}{A_{ik}}\frac{n_2}{n_i} \\
&= \left(\frac{E_{21}}{E_{ik}}\frac{A_{21}}{A_{ik}}\right) e^{-(E_2-E_i)/kT},
\end{aligned}$$

where we have assumed thermodynamic equilibrium in the second equation. The factors in parenthesis are "atomic," or quantum mechanical, and hence in principle known. Thus, relative emission line strengths provide another accurate means of determining the temperature of stellar gas. Similar considerations apply to absorption lines.

3.1.2 Absorption

Now let's turn to absorption lines. Our two-level atom can interact with an incoming photon in two ways: (1) it may absorb the photon and move from the lower 1 state to the upper 2 state, or (2) an atom in the upper 2 state may drop to the lower 1 state and emit another photon identical to the first. The first transition is what we usually think of as absorption; the second is called *stimulated emission.* Both are occurring in stars. In the presence of a radiation field of energy density (per frequency interval) u_ν, the probability per unit time of an atom in the lower state absorbing a photon and jumping to the upper state is $B_{12}u_\nu$. The probability per unit time of the reverse transition is $B_{21}u_\nu$. The B_{ij} are more Einstein coefficients, again characteristic of the atom, not the radiation field.

Thus, if n_1 and n_2 are the number densities of atoms in the lower and upper states, respectively, and taking all radiative processes into account, we can write down a simple differential equation for the rate of change of n_2:

$$\frac{dn_2}{dt} = -A_{21}n_2 + B_{12}u_\nu n_1 - B_{21}u_\nu n_2.$$

Once again, thermodynamic equilibrium simplifies things a lot, and provides us with an important connection between the Einstein coefficients. The above equation is true in all cases, and in particular in the case of thermodynamic equilibrium, where the left-hand side is zero, u_ν is given by the Planck expression

$$u_\nu \;=\; \frac{4\pi}{c}B_\nu \;=\; \frac{8\pi h\nu^3}{c^3}\left(e^{h\nu/kT}-1\right)^{-1} \;\equiv\; U_\nu\left(e^{h\nu/kT}-1\right)^{-1},$$

and n_1 and n_2 are related by the Boltzmann formula

$$\frac{n_2}{n_1} = \frac{g_2}{g_1}e^{-h\nu/kT}.$$

Thus

$$\begin{aligned} A_{21}\frac{g_2}{g_1}n_1 e^{-h\nu/kT} &= B_{12}u_\nu n_1 - B_{21}u_\nu\frac{g_2}{g_1}n_1 e^{-h\nu/kT} \\ \Rightarrow \quad A_{21}g_2\left(e^{h\nu/kT}-1\right) &= B_{12}U_\nu g_1 e^{h\nu/kT} - B_{21}U_\nu g_2. \end{aligned}$$

This latter relation must be true for any temperature T, so looking separately at the temperature-dependent and temperature-independent terms, we must have

$$\begin{aligned} \frac{A_{21}}{B_{21}} &= U_\nu \\ \frac{B_{21}}{B_{12}} &= \frac{g_1}{g_2}. \end{aligned}$$

Thus, as a ray of light of intensity I_ν passes through distance δx of the medium, the combination of the two absorption ("B") terms changes the intensity by an amount

$$\delta I_\nu = -\frac{h\nu\phi(\nu)}{c}\left(n_1B_{12} - n_2B_{21}\right)I_\nu\delta x,$$

where $\phi(\nu)$ is the line profile centered on $h\nu_0$, as before. Using the above equations to write everything in terms of A_{21}, and after a little algebra, we find

$$\delta I_\nu = -\alpha_\nu I_\nu \delta x,$$

where

$$\alpha_\nu = \frac{c^2}{8\pi\nu^2}\frac{g_2}{g_1}n_1 A_{21}\phi(\nu)\left(1 - e^{-h\nu/kT}\right)$$

is the absorption coefficient.

Note also that, through the same arguments, we must also have

$$\frac{j_\nu}{\alpha_\nu} = B_\nu.$$

3.2 Radiation Transfer in Stars

Setting aside the nitty gritty details of the previous section, we can say that, as a beam of light of intensity I_ν passes through medium, we can say that the intensity changes due to emission and absorption according to the *transfer equation*

$$\frac{dI_\nu}{dx} = -\alpha_\nu I_\nu + j_\nu.$$

The quantities j_ν and α_ν contain all necessary information about emission and absorption processes in the medium. Both are functions of density, temperature, composition, and frequency, and both, in general, are very complex. Note in particular from the preceding discussion that both are proportional to the local density of the medium. It is common to write

$$\begin{aligned} j_\nu &= \varepsilon_\nu \rho \\ \alpha_\nu &= \kappa_\nu \rho, \end{aligned}$$

where ε_ν and κ_ν are, respectively, the *emissivity* and the *opacity* of the gas.

3.2.1 Cross Sections

The absorption coefficient α_ν depends on the local density, and may be written $\alpha_\nu = \kappa_\nu\rho$, where κ_ν contains a collection of quantum-mechanical and statistical factors that describe the detailed physics of the interaction. Alternatively, we can write α_ν in terms of the local number density n. As we have just seen, α_ν has a dimension of inverse length $(1/\ell_\nu)$. Factoring out the number density, the dimension of the other factor is length squared — an area. Thus we can also write

$$\alpha_\nu = n\sigma_\nu,$$

where σ_ν is called the *cross-section* for the physical process of interest. In effect, it defines the area an absorber presents to the incoming radiation field for the given process to occur, and encapsulates the essential physics into a single number.

Cross sections give us a convenient geometric means of quantifying interaction rates. Imagine that atoms actually have some small area that they present to the radiation. A photon striking that area interacts with the atom; otherwise, no interaction occurs. We can easily calculate the probability of an interaction, as follows. Imagine again a beam of light of intensity I_ν moving through a medium of number density n and cross-section σ. Photons pass through a volume of area A perpendicular to the beam and length δx along the beam (see Maoz Fig. 3.2 and surrounding discussion). The total number of atoms in this volume $A\delta x$ is $nA\delta x$, and the total area they present to the beam is $A_{int} = n\sigma A\delta x$. Thus, as seen by photons moving through the volume, the fraction of the area filled by interacting atoms, and hence the probability of an interaction, is

$$\delta P = \frac{A_{int}}{A} = \frac{n\sigma_\nu A\delta x}{A} = n\sigma_\nu\delta x,$$

and the above relation follows.

The bottom line is that the absorption coefficient α can be rewritten in terms of the cross sections for various contributing physical processes:

$$\alpha_\nu \;=\; \sum_i n_i \sigma_{i,\nu} \;=\; \kappa_\nu \rho$$

Remember: opacity and cross-sections are just different ways of expressing the same thing.

3.2.2 Mean Free Path

We won't solve the transfer equation in detail, but we can use it to draw some important general conclusions about radiation in stars. Let's focus first on radiation moving through a purely absorbing medium ($j_\nu = 0$), so the transfer equation simplifies to

$$\frac{dI_\nu}{dx} = -\alpha_\nu I_\nu .$$

We can easily solve this by writing

$$\begin{aligned} \frac{dI_\nu}{I_\nu} &= -\alpha_\nu dx \\ \text{so} \qquad I_\nu &= I_{\nu 0}\, e^{-\int \alpha_\nu dx} \\ &= I_{\nu 0}\, e^{-\tau_\nu} . \end{aligned}$$

The incoming beam is simply attenuated as it moves through the medium. The quantity

$$\tau_\nu = \int \alpha_\nu dx$$

which determines the amount of attenuation is called the *optical depth*.

As a rule of thumb, we can think of a photon as moving through the medium until $\tau \sim 1$, at which point it is absorbed and subsequently re-emitted. The *mean free path* ℓ_ν is the distance the photon travels before this occurs. In a uniform medium, α_ν is constant with respect to x, so $\tau_\nu = \alpha_\nu \ell_\nu$ and

$$\ell_\nu = \frac{1}{\alpha_\nu} .$$

Now consider an optically thin medium, in which absorption may be neglected. In that case, the transfer equation is even simpler:

$$\frac{dI_\nu}{dx} = j_\nu ,$$

and the solution is

$$I_\nu = \int j_\nu \, dx .$$

In general, optically thin media are expected to produce emission lines (see the discussion of Kirchhoff's laws in week 2).

3.2.3 Limb Darkening and Stellar Absorption Lines

In the deep interior of the Sun, the opacity is so high that photons are effectively reabsorbed very close to where they were emitted, and light is effectively trapped. Put another way, the optical depth from any given location to the surface is very large, $\tau_\nu \gg 1$. We will discuss in more detail in a moment how solar energy works its way outward to the surface. Eventually, however, near the surface, both the opacity and the distance a photon must travel to escape decrease, and eventually we reach a point where the optical depth to the surface equals 1 — a photon has a good chance of escaping without further interaction with solar matter. The light we see therefore comes from the part of the Sun lying above the depth R_p, where

$$\int_{R_p}^{R_\odot} \alpha_\nu(r)\, dr \approx 1.$$

This region is called the *photosphere*, and given conditions in the outer layers of the Sun, it is about 500 km deep, much less than the radius of the Sun, which is why the Sun (in visible light, at least) appears to have a very sharp edge.

How deep we see into the Sun depends on the angle between our line of sight and the radial direction at the solar surface. Imagine for simplicity that α_ν is constant, α. Then if we look straight down, we see down to a depth $\ell = 1/\alpha$ (see slides3.1/13). However, if we look at an angle θ to the radial direction, the light we see started closer to the surface, at depth $\ell \cos\theta$. Since the temperature decreases with radius, we effectively see a cooler part of the Sun when we observe at an oblique angle, and the Sun looks darker. This phenomenon, called *limb darkening* is evident in slides3.1/12.

In addition, because α_ν depends on frequency, R_p does too, so the depth we can see also depends on the wavelength of the radiation. Near the center of an absorption line, where α_ν is greatest, ℓ_ν is least; conversely, near the edge, α_ν is smaller and ℓ_ν is larger (see slides3.1/14)). Thus the light we see in different parts of the line actually comes from different parts of the star. In particular, at line center we are seeing higher, cooler parts of the star, which is the real reason why the center is darker than the edges.

3.3 Free-Fall Time

See Maoz §3.1.

3.4 Hydrostatic Equilibrium

See Maoz §3.1.