PHYS 231 Lecture Notes – Week 2

Reading from Maoz (2^{nd} edition):

- **Chapter 2**

I provide here expanded coverage of material covered in class but not fully discussed in the Maoz text. For material covered in the text, I'll mostly just give some generalities and a reference. References to slides on the web page are in the format "slidesx.y/nn," where x is week, y is lecture, and nn is slide number.

2.1 Fluxes and Filters

No detector can cover the entire electromagnetic spectrum, so the flux obtained by integrating over all frequencies or wavelengths (called the bolometric flux) is not really measurable. Fortunately in many cases most of the energy is emitted in the range of a single instrument, so the Stefan-Boltzmann law quoted above is still approximately correct if we use the detected flux rather than the total one.

In fact, rather than always obtaining detailed spectra of stars, astronomers often find it convenient to measure the light received through a number of standard filters attached to the telescope (see for example slides 2.1/9 and 10). The principal reason for doing this is that it is much faster to observe an object through a small number of broad-band filters than it is to split the light up into hundreds or thousands of wavelength bins to obtain a complete spectrum. Astronomers do work with detailed spectra, but taking such a spectrum of every star is impractical.

Each of the filters shown on the slides effectively defines a slice of the spectrum. Their names describe roughly the color of light they transmit—B is blue, V is the middle of the visible spectrum, R is red, U is ultraviolet, etc. These days many instruments have their own custom designed filters, so the terminology is not universal, but conversions between the different filter sets can easily be made.

Let's say we are looking at a star through the V filter. Then the flux we observe is really

$$f_V = \int V(\lambda) f_\lambda \, d\lambda$$

so the integral is not really from 0 to ∞ any more, because of the filter. Similarly, we talk about V luminosity, L_V, etc. The inverse-square law continues to hold:

$$f_V(D) = \frac{L_V}{4\pi D^2}.$$

Filters are useful because they let us quantify the color of a source. Imagine again a blackbody spectrum. Because the spectrum depends on just 1 parameter – the temperature T—you should be able to convince yourself that if I measure the fluxes at two wavelengths, the ratio of those fluxes will uniquely determine T. A star with a lot more blue flux than red flux is hotter than one with more red than blue. Astronomers do essentially the same thing with filters to define stellar colors.

But there's a operational catch. Astronomers generally don't work directly with fluxes. Instead, they express stellar brightness in terms of an ancient construct called the magnitude scale. Everything we just said still applies, but we need to define some terminology first.

2.2 Magnitudes and Colors

Ancient astronomers didn't measure stellar fluxes—the technology to do that only became available in the 19th century. Instead, following the system invented by the Greek astronomer Hipparchus, they ranked the several thousand stars visible with the naked eye by brightness, and that ranking was called the star's *magnitude.* First magnitude stars were the brightest, sixth magnitude stars the faintest visible.

When astronomers finally became able to quantify fluxes, they found that (i) the magnitude scale is logarithmic—an increase of one magnitude corresponds to a decrease in flux by a constant factor, and (ii) first magnitude stars are about 100 times brighter than sixth magnitude stars. The modern magnitude scale retains the ranking, the logarithmic scaling, and this calibration, and defines the magnitude of a star (in the V band, say) as

$$m_V = -2.5 \log_{10} f_V + \text{constant},$$

where the constant is a normalization such that certain standard stars have specific magnitudes. For example, the star Vega was originally defined to have a V magnitude of 0.0 (the scale has shifted since, so Vega now has $m_V = 0.03$). The brightest star in the sky is Sirius, with $m_V = -1.47$.

In practice, the constant often is unimportant, since we are usually interested in ratios of fluxes, corresponding to differences in magnitudes, so the constant cancels. A better definition of the scale is that if two stars have fluxes (in some filter) f_1 and f_2 and magnitudes m_1 and m_2, then we can say

$$m_2 - m_1 = -2.5 \log_{10} \left(\frac{f_2}{f_1} \right).$$

Some more terminology. It is very common in observational astronomy to denote the magnitude in the X band (m_X) simply as X, so V is the same as m_V, B is m_B, etc. We mentioned earlier that the ratio of fluxes at two wavelengths is sufficient to determine the temperature, and hence the color of a blackbody spectrum. Following that reasoning, since a ratio of fluxes is a difference in magnitudes, the *color* of an astronomical object is the the difference in its magnitudes through two different filters, e.g.

$$B - V \;=\; m_V - m_B \;=\; -2.5 \log_{10} \left(\frac{f_B}{f_V} \right).$$

Many colors can be defined, and different choices may be very useful in different circumstances, but some common ones are $B - V, U - B, V - I$, and their equivalents in other filter schemes.

Just remember, B and V are fluxes, and $B - V$ is a color. For a perfect blackbody, $B - V$ is equivalent to temperature, and the same is approximately true for real stars too. Note that, since color is a difference in magnitudes, or a ratio of fluxes, it is *independent* of the distance to the source.

Finally, since the magnitudes we have just described are really fluxes, or apparent brightness, they are generally referred to as *apparent magnitudes.* Astronomers also express luminosities using magnitudes, as follows. If we imagined seeing all stars at a standard distance away, then their light would all be diminished by the same amount by the inverse square law, and so the fluxes we measured would be proportional to the luminosities. Astronomers conventionally choose that standard distance to be 10 pc. There is no particular reason for that choice—it is completely arbitrary. The apparent magnitude of a star seen at a distance of 10 pc is called its *absolute magnitude.* Absolute magnitude, usually denoted M_V in the V band, is equivalent to luminosity.

Comparing apparent and absolute (V) magnitudes for a star at distance D, we have

$$\begin{aligned} m_V - M_V &= -2.5\log_{10}\left(\left[\frac{L_V}{4\pi D^2}\right] \Big/ \left[\frac{L_V}{4\pi(10\text{pc}^2)}\right]\right) \\ &= 2.5\log_{10}\left(\frac{D}{10\text{pc}}\right)^2 \\ &= 5\log_{10} D(\text{pc}) - 5. \end{aligned}$$

This is the inverse square law expressed in magnitudes. The quantity on the left hand side is often referred to as a star's *distance modulus.*

2.3 The Color–Magnitude Diagram

Now that we can measure magnitudes and colors for stars, we can begin to look for correlations among them. The result is a *color–magnitude diagram (CMD)*, several of which are shown in slides 2.1/52–55. The horizontal axis is color, the vertical axis is apparent magnitude. For most of the examples shown in class, the color is $B - V$, which is equivalent to temperature increasing right to left, and the diagram is plotted for stars in star clusters, so apparent magnitude is equivalent to luminosity, so the plot can equivalently be viewed as plotting the fundamental stellar parameters T and L. With the latter axes, the diagram is called a *Hertzsprung–Russell (H–R)* diagram, after the two astronomers who first made such plots about a century ago. H–R diagrams are widely used by theorists in discussions of stellar structure and evolution. CMDs are what observers actually measure.

Note that stars aren't distributed randomly across these diagrams. Most (about 90 percent) lie on a broad band running from the upper left (bright and blue) to the lower left (faint and red). This region is called the *main sequence* and (as we will see) it is where stars lie when they first begin to fuse hydrogen in their cores. Many main sequence stars have measured masses, using the techniques described earlier. We find that the main sequence is first and foremost a mass sequence—the most massive stars are at the top, the lowest-mass stars at the bottom. Indeed, if all stars had identical composition, the main sequence would be a narrow line stretching across the H-R diagram. The observed spread in the sequence is due primarily to composition differences.

The rest of the diagram is not empty. There is a sequence of bright red stars at top right and another group of faint blue-white stars at lower left. Using the relation $L \propto R^2T^4$ described earlier, we can estimate stellar radii, as indicated in slide 2.2/45, and these estimates explain the names for these groups of stars. The bright red stars are called *red giants*, and the faint blue-white stars are *white dwarfs.*

All this structure in the H-R diagram means that we have something to explain. The theories of stellar structure and evolution are how we will do that.

2.4 Mass and Luminosity

We can get a basic understanding of why stars must evolve—change their properties in a systematic way as time goes on—from a very simple argument. When we plot stellar luminosities against stellar masses, we find a clear mass–luminosity relation (slides2.1/56 and 57). As discussed in class, the details depend on the interior structure of the star, but over a broad range of masses we have a simple scaling

$$L \propto M^4.$$

Now let's ask why stars shine. What is responsible for the luminosity we see? The answer is that main sequence stars are fusing hydrogen into helium in their cores, and the energy released ultimately leaves the surface as visible light. Stars are converting mass into energy, and the luminosity we see is proportional to the rate at which mass is being consumed.

In this simple view, a star is fuel for nuclear fusion. When a significant amount of the mass of the star has been consumed (actually, about 10 percent), the star has effectively run out of fuel and something new has to happen. Again, that something is what stellar evolution theory is all about. Dividing the amount of fuel (M) by the rate of consumption (L) and not worrying about the many details, we can define a stellar *main sequence lifetime*

$$\begin{aligned} t_{MS} &\sim \frac{M}{L} \\ &\sim \frac{M}{M^4} \\ &\sim M^{-3}. \end{aligned}$$

Thus, not only can we say (i) stars must evolve in time, but (ii) the most massive ones evolve fastest! We'll see later that the main-sequence lifetime of the Sun is about 10 billion years, so we can write, approximately,

$$t_{MS}(M) \approx 10^{10} \left(\frac{M}{M_\odot}\right)^{-3} \text{ yr}.$$

2.5 Cluster Distances and Ages

We can use observations of color-magnitude diagrams to determine both the distances to and the ages of star clusters. Star clusters are important to astronomers because the stars all lie in the same relatively small region of space and hence essentially at the distance from us, and all formed at almost the same time out of the same material, and so have virtually the same composition. This makes them ideal laboratories for studying stellar physics.

Imagine we have two CMDs: (1) an observed cluster CMD, with apparent magnitude on the vertical scale (slides2.2/7), and (2) a "standard" CMD showing a well defined main sequence (slides2.2/8), obtained from theoretical models calibrated against many local cluster observations and having absolute magnitude on the vertical scale. Both have color on the horizontal axis. If we assume that the theoretical model is a good description of the observed cluster, the only freedom we have is to choose the distance, which amounts to sliding one diagram vertically up or down until it matches the other (slides2.2/9). (Remember that the color is a ratio of fluxes and is therefore independent of distance.) Since

$$m - M = 5\log_{10} D \text{ (pc)} - 5$$

we can easily determine D by this process.

Note that this is a statistical generalization of the standard candle idea. Instead of choosing one star, assuming it is on the main sequence, and using that information to determine distance, by matching the full main sequences we are effectively obtaining a best-fit distance for the entire cluster.

Once we know the distance we can also estimate the age. Recall our earlier discussion of the mass-luminosity relation, where we concluded that stars must evolve, and that a star's main-sequence lifetime is

$$t_{MS}(M) \approx 10^{10} \left(\frac{M}{M_\odot}\right)^{-3} \text{ years}.$$

This means that the top of the main sequence "peels off" as high-mass stars turn into something else. We can readily identify the most massive star still on the main sequence. This is a so-called *main sequence turn-off*, and represents the star whose main-sequence lifetime corresponds to the present age of the cluster. We can determine the turn-off just by looking at the CMD and, once we have calibrated to find D, we can read its absolute magnitude from the vertical scale, then convert that to a luminosity (knowing the absolute V magnitude of the Sun is 4.8), and then use the mass radius luminosity to get a mass and hence an age.

Note that there is one caveat in this process. Many clusters contain stars known as *blue stragglers*, which lie on the main sequence above where the turnoff should be, and hence appear to be younger than the rest of the cluster. You can see several in the Hyades and M3 CMDs in the slides (slides2.2/7 and 10). In the absence of other information, knowing which stars to ignore can be something of an art. In the case of the Hyades, the upturn in the theoretical models at the top of the main sequence represents the true turnoff, and should be used in age estimations.

2.6 Stellar Spectra

We haven't talked at all yet about spectral lines in the Sun and other stars. Clearly the nice clean blackbody spectra we were discussing previously are substantially altered in reality by spectral lines — see slides2.2/12. All of the spectra in that figure shoe prominent *absorption lines*, although the lines themselves vary a lot from star to star. The Sun, which we can observe in great detail has thousands of dark *Fraunhofer* lines in its spectrum.

2.6.1 Emission and Absorption Lines

It will take us too far afield to go into detail on atomic structure and spectral lines, but let's take a few moments to review what you should already know from other physics courses. It has been known for some 200 years that when gases are heated they produce an *emission spectrum* of sharp spectral lines whose colors are characteristic of the material (slides2.2/13,16). Furthermore, when a continuous spectrum like a blackbody passes through a cool sample of the same gas, an absorption spectrum is seen, with dark lines at the same locations as the bright lines in the emission spectrum (slides2.2/15). These connections are summarized in Kirchhoff's laws, which are illustrated in slides2.2/17:

- A hot, dense solid, liquid or gas produces a continuous spectrum.
- A hot, low-density gas produces a bright-line or emission spectrum.
- When a source of a continuous spectrum is viewed through a cool gas, a dark-line or absorption spectrum is seen.

Thus, long before anyone knew any atomic physics, spectroscopists were aware that emission and absorption lines must have the same underlying physical cause, and that they represented a unique "fingerprint," allowing them to identify elements and molecules in the light from distant objects (like stars). With the development of quantum physics and its application to atomic structure in the early 20th century, scientists finally understood the origin of spectral lines and the connection between them. Briefly, electrons in atoms can exist only in certain well-defined energy states (E_i, say), and when the atom shifts between one state (i) and another (j) it can do so by emitting or absorbing a photon of energy corresponding precisely to the energy difference:

$$h\nu = |E_j - E_i|.$$

Since the energy levels are quantized, the photon energies are too, so the origin of and connection between emission and absorption spectra is clear. Slides2.2/20 illustrates the connection between the well-known Bohr model of the hydrogen atom and the line spectrum of hydrogen.

2.6.2 Stellar Spectral Classification

Now back to observed stellar spectra. By the end of the 19th century astronomers had amassed tens of thousands of stellar spectra, and had begin to classify stars according to the lines they observed. Indeed, one well-known element — helium — was discovered spectroscopically on the Sun (hence the name, from the Greek *helios*) before it was found on Earth. Since no-one knew the origin of the lines, the classification basically assumed that the strengths of the lines indicated the composition of the star. Stars were assigned *spectral types* A, B, C,... based primarily on the strengths of the visible hydrogen lines (the Balmer sequence in slides2.2/20), with additional criteria based on other atomic and molecular spectral features.

Once the structure of the atom was understood, it became clear that the differences among stars were due primarily to temperature (see below), not to composition. Stars all have basically the same composition (91% hydrogen, 9% helium; see slides2.2/21), although small but important composition differences do exist. Several of the spectral classes were discarded, and the remainder, now ranked in order of decreasing temperature, are listed in slides2.2/23,25. From hot to cool, the standard spectral classes are O, B, A, F, G, K, M (Oh Be A Fine Guy/Girl, Kiss Me is the standard mnemonic). This terminology is widely used throughout astronomy, which is the main reason to mention it here, even though spectral types are no longer used as a primary means of quantifying stellar properties. We could go through the entire course without ever mentioning what an O star is, but all astronomers know that O and B stars are hot, bright, blue, and short-lived, while K and M stars are the opposite. The Sun, by the way, is of type G2 (the 2 is part of a decimal subdivision of the letter types, where 0–9 represent decreasing temperature).

2.7 Some Thermodynamics and Statistical Mechanics

In order to discuss how stellar spectra depend on temperature, we need to say something about temperature determines the speeds, excitation levels, and ionization states of atoms in a star. There are many different physical processes going on, involving collisions among atoms and electrons, and interactions between matter and radiation. All of these processes transform energy from one form to another, and in principle trying to follow them all can get very complicated. Fortunately, there is a huge simplifying assumption we can make — *thermodynamic equilibrium.*

The idea is that, in a completely closed system, we expect all processes to come into equilibrium, in the sense that all physical properties of the material — temperature, density, atomic excitation state, ionization fraction — don't fluctuate in time. In other words, although atoms are banging into one another and bumping electrons up and down the atomic energy states, we can assume that for every process causing a transition from 1 to 2, there is another from 2 to 1, and they are *in balance*, meaning that the total numbers of atoms in state 1 or state 2 stay constant. The same applies to the speeds of atoms in the gas, and to interactions between atoms and radiation. There is a well-defined temperature T, and this in essence controls the equilibrium of all processes. In fact, stars are not closed systems — energy is constantly leaking out, and their properties depend on location in the star — but these deviations deviations from perfect thermodynamic equilibrium are generally small, and we can assume *Local Thermodynamic Equilibrium* in discussing the properties of the gas.

2.7.1 The Maxwell-Boltzmann Distribution

With this assumption we can use the techniques statistical mechanics to determine the most probable distribution of physical properties. For example, in an ideal classical (non-relativistic) gas made up of particles (atoms) of mass m, the temperature T may be defined in terms of the mean kinetic energy of the gas particles:

$$\langle \tfrac{1}{2}mv^2 \rangle = \tfrac{3}{2}kT.$$

Statistical mechanics tells us how the speeds v are distributed. The probability of an atom having speed v in the range $[v, v+\delta v)$ is $p(v)\delta v$, where

$$p(v) = \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2\, e^{-mv^2/2kT}.$$

The first factor simply ensures that $\int p(v)\,dv = 1$. Even though atoms are bouncing around and colliding with one another and an individual atom may change its speed billions of time per second, *on average* the distribution of speeds stays constant. The above expression is called the *Maxwell-Boltzmann* distribution.

Note that since atoms are moving in all directions with many different speeds, the light they emit or absorb will be Doppler shifted slightly due to their motion relative to an observer. Thus, for an example, a emission line that is very narrow in the rest frame of an atom will be broadened due to the thermal motion of the atoms (see slides2.2/27). Since the rms speed is

$$v_{rms} \;=\; \langle v^2 \rangle^{1/2} \;=\; \sqrt{\frac{3kT}{m}},$$

the line broadening will be on the order of

$$\left\langle \frac{\Delta\lambda}{\lambda} \right\rangle \;\sim\; \frac{v_{rms}}{c} \;\sim\; \sqrt{\frac{3kT}{mc^2}}.$$

Thus measuring the widths of spectral lines gives us direct information on the temperature of the gas producing them.

2.7.2 The Boltzmann Formula

Similar reasoning applies to the energy distribution of electrons in atoms. Imagine two electron states A and B, with $E_2 > E_1$ (slides2.2/31). As atoms collide and energy is exchanged, the electron will be bumped up and down between the states many times, but averaged over many atoms, statistical mechanics tells us that the ratio of the number of atoms in the upper state, n_2 to the number in the lower state, n_1 is given by the *Boltzmann formula*

$$\frac{n_2}{n_1} \;=\; \frac{g_2}{g_1}\, e^{-(E_2-E_1)/kT}.$$

Here, g_2 and g_1 are the *degeneracies* of the two states, which simply means the number of different quantum states that have the same energy, or the number of different ways the electron can have that energy. The degeneracy of states in hydrogen can be directly calculated from quantum mechanics: $g_n = 2n^2$, the n^2 coming from all allowed angular momentum states and the 2 from the two spin states of the electron.

Note that, as $T \to 0$, the exponential term goes to zero and $n_2 = 0$ — in other words, all atoms are in the lower state. This immediately explains why low-temperature M stars don't have any hydrogen absorption lines. The visible (Balmer) hydrogen lines correspond to transitions starting at the first excited state ($n = 2$; see slides2.2/20). But for such a transition to occur, we must have some atoms in the $n = 2$ state. A transition between $n = 1$ and $n = 2$ has energy $E_{21} = E_2 - E_1 = 10.2$ eV. Thus, for this transition,

$$\frac{n_2}{n_1} = 4\, e^{-E_{21}/kT}.$$

For $T = 5000$ K, this ratio is 2.1×10^{-10}, so negligibly many hydrogen atoms are in the excited state. Increasing T to 10,000 K increases the ratio by a factor of more than 10^4, to 2.8×10^{-5} — still a small number, but large enough that enough hydrogen is in the excited state to create a strong line.

2.7.3 The Saha Equation

As $T \to \infty$ in the Boltzmann formula, the exponential factor goes to 1, so

$$\frac{n_2}{n_1} = \frac{g_2}{g_1},$$

meaning that, in the absence of other effects, all states would be populated according to their degeneracy. However, as the temperature increases, it becomes more and more likely that the atom will be *ionized.*

An atom (or ion) becomes ionized then a bound electron gains enough energy to escape. This can occur radiatively (via the absorption of a photon) or collisionally (when two atoms run into one another). The reverse process is called *recombination*, where an electron and an ion encounter one another, lose some energy in the form of a photon, and become bound. This is a reversible process:

$$X_i \longleftrightarrow X_{i+1} + e,$$

where X_i represents the i-th ionization state of element X. Spectroscopists often use the notation XI (roman numeral I to refer to the neutral atom, XII the singly ionized atom, etc., so some concrete examples might be

$$\begin{aligned} OI &\longleftrightarrow OII + e \\ NII &\longleftrightarrow NIII + e \\ HI &\longleftrightarrow HII + e. \end{aligned}$$

In ionization equilibrium, the forward and backward rates are equal, and statistical mechanics tells us that the number densities of state i, state $i + 1$, and electrons must satisfy the *Saha equation*

$$\frac{n_{i+1} n_e}{n_i} = \frac{2}{\Lambda^3} \frac{g_{i+1}}{g_i}\, e^{-\chi_i/kT}.$$

Here, g_i is the degeneracy of ionization state i, χ is the energy needed to ionize state i to state $i + 1$, and

$$\Lambda = \left(\frac{h^2}{2\pi m_e kT}\right)^{1/2}$$

is a length — in fact, the de Broglie wavelength of an electron. For typical "stellar" temperatures,

$$\Lambda = 7.5 \times 10^{-10} \text{ m} \left(\frac{T}{10^4 \text{ K}}\right)^{-1/2}.$$

Let's focus on the ionization state of hydrogen. Let the total number density of hydrogen nuclei (atoms plus protons), hydrogen atoms, protons, and electrons be n, n_H, n_p, and n_e, respectively, and define the *ionization fraction* as

$$X = \frac{n_p}{n}.$$

Clearly, $n = n_h + n_p$ and, if all electrons were produced by ionization of hydrogen (not a bad assumption for most stars), $n_e = n_p$. In that case, $g_i = 4$ (2 proton spin states times 2 electron spins), $g_{i+1} = 2$) (2 proton spin states), and the Saha equation becomes (with $n_H = (1-X)n$)

$$\frac{X^2}{1-X} = \frac{1}{n\Lambda^3}\ e^{-\chi/kT},$$

where $\chi = 13.6$ eV. Thus, although we can't write down an explicit solution for X in terms of n and T, once n and T are specified, solving for X is just a matter of solving a quadratic equation

$$X^2 + AX - A = 0,$$

where A is the right-hand side of the above equation.

Slides2.2/34 shows the solution to this equation as a function of T for a range of choices of hydrogen density (left to right): $n = 10^6, n = 10^9, n = 10^{20}, n = 10^{24}, n = 10^{26}$, and $n = 10^{28}$ m^{-3}, corresponding to densities found in interstellar space, near the solar surface, and in the solar interior. Note that (unsurprisingly) X increases as T increases, and (less obviously) decreases as n increases, because recombination is faster at higher densities. For densities typical of the solar interior close to the surface, $X > 0.5$ for $T > 17000$ K. The outer layers of hot O and B stars are almost completely ionized, which explains why no hydrogen absorption lines are seen — there is no atomic hydrogen to produce them.

If we simply estimate the temperature T at which the typical atomic energy $\frac{3}{2}kT$ equals the ionization energy of hydrogen, 13.6 eV, we find $T = 1.5 \times 10^5$ K, a temperature consistent with the above discussion of the Boltzmann formula (in that the exponential term is close to 1), but much hotter than any known star. Why then, is the hydrogen near the solar surface ionized? The answer is that the ionization fraction is determined by a balance between ionization and recombination. As the density decreases, recombination becomes less effective, and the equilibrium moves toward full ionization. In the outer layers of the Sun, ionization occurs due to relatively rare interactions that lie far out on the high-energy exponential tail of the Maxwell-Boltzmann (or Planck) distribution, with $E \gg kT$, but if recombination is ineffective in opposing this process, they can nevertheless lead to a large ionization fraction.