PHYS 231: Introductory Astrophysics

Winter 2020

Homework #6 solutions

1. (a) For a circular orbit at radius R, the orbital speed is $V = \sqrt{GM/R}$ and so the angular momentum per unit mass is $VR = \sqrt{GMR}$. Hence, for accretion rate $\dot{M}$, the angular momentum L of the neutron star increases at a rate $\dot{L} = \sqrt{GMR}\,\dot{M}$.

   (b) Given $L = \frac{2}{5}MR^2\Omega$, then, assuming constant M and R, $\dot{L} = \frac{2}{5}MR^2\dot{\Omega} = \sqrt{GMR}\,\dot{M}$, so $\dot{\Omega} = \frac{5}{2}\dot{M}\sqrt{G/MR^3}$. Putting in the numbers, we find $\dot{\Omega} = 2.40 \times 10^{-12}\,\mathrm{rad\,s^{-2}}$.

   (c) Neglecting changes in M and R, we have $\Omega(t) = \Omega_0 + \dot{\Omega}t$, assuming that $\Omega_0 = 2\pi\,\mathrm{rad\,s^{-1}}$ is in the same sense as the acceleration from the accretion flow. The rotation speed reaches $\Omega = 2000\pi\,\mathrm{rad\,s^{-1}}$ when $t = (\Omega - \Omega_0)/\dot{\Omega} = 83\,\mathrm{Myr}$.

2. Rearranging Equations (5.7) and (5.8) in the book, it is straightforward to show that the Jeans length and mass associated with gas density ρ and temperature T are

$$
\begin{aligned}
M_J &= \frac{9}{2\pi^{1/2}}\left(\frac{kT}{G\bar{m}}\right)^{3/2}\rho^{-1/2} \\
R_J &= \frac{3}{2\pi^{1/2}}\left(\frac{kT}{G\bar{m}}\right)^{1/2}\rho^{-1/2}.
\end{aligned}
$$

   (a) A molecular cloud core with temperature $T = 20$ K, $X = 0.7$, $Y = 0.3$, and number density $n_H = 10^6\,H_2$ molecules per cubic centimeter has $\rho_H = 2m_pn_H$, $\rho_{He} = \rho_H(Y/X)$, $n_{He} = \rho_{He}/4m_p$, and $n = n_H + n_{He} = n_H(1 + Y/2X)$, assuming composition of purely molecular hydrogen and atomic helium. Thus $\bar{m} = \rho/n = 4m_p/(2X + Y) = 2.35\,m_p$, $\rho = \bar{m}n = \bar{m}n_H\,(1 + Y/2X) = 2.78 \times 10^{-15}\,\mathrm{kg\,m^{-3}}$, and

$$M_J = 0.63\,M_\odot, \qquad R_J = 0.013\,\mathrm{pc}.$$

   (b) A cool HI cloud of temperature $T = 250$ K and number density $n_H = 30\,H$ atoms per cubic centimeter has $\rho_H = m_pn_H$, $\rho_{He} = \rho_H(Y/X)$, $n_{He} = \rho_{He}/4m_p$, and $n = n_H + n_{He} = n_H(1 + Y/4X)$, now assuming composition of purely atomic hydrogen and helium. Thus $\bar{m} = \rho/n = 4m_p/(4X + Y) = 1.29\,m_p$, $\rho = \bar{m}n = \bar{m}n_H\,(1 + Y/2X) = 7.17 \times 10^{-20}\,\mathrm{kg\,m^{-3}}$, and

$$M_J = 1.77 \times 10^4\,M_\odot, \qquad R_J = 15.9\,\mathrm{pc}.$$

   (c) A region of warm ionized medium with temperature $T = 10^4$ K and density 0.1 hydrogen masses per cubic centimeter has $\rho = 10^5 m_p\,\mathrm{kg\,m^{-3}}$, $\rho_H = X\rho$, $\rho_{He} = Y\rho$, $n = 2\rho_H/m_p + \rho_{He}/4m_p$, assuming fully ionized hydrogen and atomic helium. Thus $\bar{m} = \rho/n = m_p/(2X + Y/4) = 0.68\,m_p$, $\rho = 1.67 \times 10^{-22}\,\mathrm{kg\,m^{-3}}$, and

$$M_J = 2.43 \times 10^8\,M_\odot, \qquad R_J = 2.86\,\mathrm{kpc}.$$

3. In the frame rotating at angular speed $\Omega = \sqrt{GM/R^3}$ there are three forces acting on the test particle: the gravity of masses M and m, and the centrifugal force $\Omega^2 x$. The Lagrange points are where the three forces balance. It is simplest to look separately at the three regimes (a) $x > R$, (b) $0 < x < R$, and (c) $x < 0$.

   (a) In this case, both gravitational forces are toward negative x, while the centrifugal force is toward positive x, so at the point of equilibrium,

$$-\frac{GM}{x^2} - \frac{Gm}{(x-R)^2} + \Omega^2 x = 0.$$

   Note that if we just ignored the Gm term we would find $x = R$, which can't be correct, so let's set $x = R + \delta$, where δ is assumed to be small, and then expand everything using the binomial theorem, to first order:

$$-\frac{GM}{R^2}\left(1 - \frac{2\delta}{R}\right) - \frac{Gm}{\delta^2} + \frac{GM}{R^3}(R + \delta) = 0.$$

   We see that, as expected, the GM/R^2 terms cancel, leaving us with

$$\frac{3M\delta}{R^3} = \frac{m}{\delta^2},$$

   so

$$\delta = \left(\frac{m}{3M}\right)^{1/3} R.$$

   This is the L_2 Lagrange point.

   (b) Similarly, for $0 < x < R$, we have

$$-\frac{GM}{x^2} + \frac{Gm}{(x-R)^2} + \Omega^2 x = 0.$$

   Again writing $x = R + \delta$ and expanding, we find

$$-\frac{GM}{R^2}\left(1 - \frac{2\delta}{R}\right) + \frac{Gm}{\delta^2} + \frac{GM}{R^3}(R + \delta) = 0,$$

   so

$$\frac{3M\delta}{R^3} = -\frac{m}{\delta^2},$$

   so

$$\delta = -\left(\frac{m}{3M}\right)^{1/3} R.$$

   This is the L_1 Lagrange point.

   (c) For $x < 0$, the equilibrium point is now given by

$$\frac{GM}{x^2} + \frac{Gm}{(x-R)^2} + \Omega^2 x = 0.$$

   Ignoring the Gm term would give us the solution $x = -R$, which is approximately correct, so let's refine that estimate by writing $x = -R + \delta$ and expanding as before. The leading GM terms again cancel and we are left with

$$\frac{GM}{R^2}\left(1 + \frac{2\delta}{R}\right) - \frac{Gm}{4R^2} + \frac{GM}{R^3}(-R + \delta) = 0,$$

where we have ignored a term involving $\delta \times m$. After cancellation we are left with

$$\frac{3M\delta}{R^3} = -\frac{m}{4R^2},$$

so

$$\delta = -\left(\frac{m}{12M}\right) R.$$

This is the L_3 Lagrange point.

4. (a) For a period of 20 days, I estimate from the diagram an absolute magnitude of $M \approx -4.7$. If the apparent magnitude is $m = 19$, then the distance is $D = 10^{(m-M)/5+1}\,\mathrm{pc} = 550\,\mathrm{kpc}$.

   (b) The distance at which HST could just detect at apparent magnitude $m = 27$ a supernova with absolute magnitude $M = -19.6$ is $D = 20.9\,\mathrm{Gpc}$.

5. (a) From top to bottom, at $R = 20$ kpc, I estimate $v \approx 347$, 292, 257, 249, 245, and 198 km/s. Hence the masses are $M \sim Rv^2/G = 5.6, 4.0, 3.1, 2.9, 2.8$ and $1.8 \times 10^{11}\,M_\odot$.

   (b) The mass inside radius R is

$$M(R) \;=\; \int_0^R 4\pi r^2 \rho(r)\,dr \;=\; \int_0^R 4\pi A\,dr \;=\; 4\pi AR.$$

   Since $M = Rv_c^2/G$, it follows that $A = v_c^2/4\pi G$.

6. For completely ionized pure hydrogen, the Eddington luminosity is

$$L_{edd} \;=\; \frac{4\pi c G M m_p}{\sigma_T},$$

   where M is the mass of the source, m_p is the mass of a proton, c is the speed of light, and σ_T is the Thomson cross section.

   (a) For $M = 10^9\,M_\odot$, $L = 1.3 \times 10^{40}$ W $= 3.2 \times 10^{13}\,L_\odot$, or 13% of the stated bright AGN luminosity.

   (b) For efficiency η and mass accretion rate $\dot{M}$, the luminosity is (by definition) $L = \eta \dot{M} c^2$. If L is the luminosity from part (a) and $\eta = 0.057$, it follows that $\dot{M} = L/\eta c^2 = 39\,M_\odot/\mathrm{year}$.

   (c) If the black hole is always accreting at the Eddington rate, we have

$$\begin{aligned}\frac{dM}{dt} &= \frac{L_{edd}}{\eta c^2} \\ &= \frac{4\pi G m_p}{\sigma_T \eta c}\,M \\ &= \frac{M}{\tau},\end{aligned}$$

   where $\tau = \sigma_T \eta c / 4\pi G m_p = 25.7\,\mathrm{Myr}$. Hence the solution is $M(t) = M_0 e^{t/\tau}$, so for $M_0 = 1000\,M_\odot$, $M(t) = 2 \times 10^9\,M_\odot$ when $t = 14.5\,\tau = 370$ Myr.

7. (a) From the data, the average redshift is $\langle z \rangle = 0.1196$, corresponding to a mean recession velocity of $V = c\langle z \rangle = 3.587 \times 10^4$ km/s.

   (b,c) The standard deviation of the redshifts is $\sigma_z = 0.000911$, corresponding to a 1-D velocity dispersion in the cluster of $\sigma = c\sigma_z = 0.000911 = 273.3$ km/s.

   (d) The angular radius of the cluster as defined is $674''$. At a distance of 500 Mpc, this corresponds to a radius of $R = 1.63$ Mpc.

   (e) The virial estimate of the mass of the cluster is $3R\sigma^2/G = 2.55 \times 10^{14}\ M_\odot$.

   (f) Based on the mean recession velocity V and the given distance $D = 500$ kpc, the best estimate of the Hubble constant is $H_0 = V/D = 71.7$ km/s/Mpc.

8. (a) At redshift $z = 0.1$, the recession velocity is $V = cz = 3 \times 10^4$ km/s. The distance then is $D = V/H_0 = 429$ Mpc.

   (b) If the radius is $R = 15$ kpc, then the angular diameter is $\theta = 2R/D = 14.4''$.

   (c) The redshift corresponding to a peculiar velocity of $v_p = 600$ km/s is $z = v_p/c = 0.002$. That would introduce a fractional error $\Delta z/z = 0.002/0.1 = 0.02$ into the estimate of the redshift.