PHYS 231: Introductory Astrophysics

Winter 2020

Homework #5

(Due: March 2, 2020)

Each problem is worth 20 points.

1. A neutron star has mass $1.3\,M_\odot$ and radius 12 km.

   (a) What is its surface gravity, in Earth gravities?

   (b) What is its mean density?

   (c) Approximating it as a homogeneous sphere, what is its gravitational potential energy?

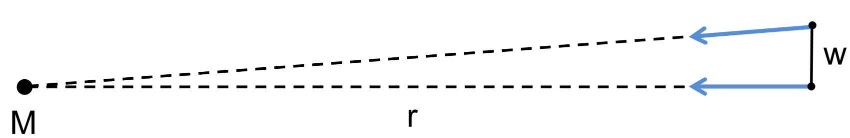
   (d) If the neutron star is made entirely of neutrons, and the mean interparticle separation in a gas of number density n is $\ell \approx n^{-1/3}$, how far apart are typical neutrons from one another? Compare this separation with the $\sim 10^{-15}$ m characteristic radius of a neutron.

   (e) Suppose the neutron star is rotating so fast that its equator was moving at exactly the speed of light (not actually possible!). What would be its rotation period?

2. (a) Imagine you are 2 m tall and falling feet first into a $10\,M_\odot$ black hole. What would be the differential (tidal) acceleration between your head and your feet as your feet crossed the event horizon? Approximate the gravitational acceleration as the usual Newtonian expression $\mathbf{a} = -GM\mathbf{r}/r^3$.

   (b) As the black hole mass increases, the differential acceleration at the event horizon decreases. Calculate the mass of the black hole for which the differential acceleration between your feet and your head at the event horizon would be a gentle (and probably unnoticeable) $0.001\,g$.

   (c) Derive an expression for the *transverse* tidal acceleration produced on your body, as follows. Again assume a Newtonian gravitational acceleration and consider the relative accelerations of two points separated by distance w perpendicular to the radial direction, as illustrated in the diagram below, with $w \ll r$. Compute the component of the relative acceler-



   ation in the transverse (vertical) direction, Δa_t, and compare it to the radial tidal acceleration from part (a).

3. Imagine two identical objects, each of mass M, orbiting one another on circular orbits, with separation a.

(a) Show that the total orbital energy of the binary system is

$$E = -\frac{GM^2}{2a}.$$

Hence show that

$$\frac{dE}{dt} = \frac{GM^2}{2a^2}\frac{da}{dt}.$$

(b) The system loses energy due to gravitational radiation at a rate

$$\dot{E}_{GW} = -\frac{64G^4M^5}{5c^5a^5}.$$

By equating the above expressions, show that the separation a satisfies the differential equation

$$\frac{da}{dt} = -\frac{128G^3M^3}{5c^5}a^{-3}.$$

(c) Initially ($t = 0$) the binary has separation a_0. Solve the differential equation to find the merger time t_{merge} at which $a = 0$.

(d) Evaluate t_{merge} for (i) two $1\,M_\odot$ white dwarfs with $a_0 = 0.01$ AU; (ii) two $30\,M_\odot$ black holes with $a_0 = 0.1$ AU.

4. Photon orbits are particularly easy to calculate in the Schwarzschild metric. They have $ds = 0$, so

$$\left(1 - \frac{r_s}{r}\right)c^2dt^2 = \left(1 - \frac{r_s}{r}\right)^{-1}dr^2,$$

where $r_s = 2GM/c^2$. For an outgoing photon (with $r > r_s$),

$$\frac{dr}{dt} = c\left(1 - \frac{r_s}{r}\right).$$

(a) A photon is emitted at time $t = 0$ at radius $r_0 > r_s$ and travels radially outward. Solve the above equation to show that its radius r at any subsequent time t is given by

$$ct = (r - r_0) + r_s\ln\left(\frac{r - r_s}{r_0 - r_s}\right).$$

(b) Show that, for $r - r_0 \gg r_s$,

$$t \approx \frac{r - r_0}{c}.$$

(c) Show that, for any r, $t \to \infty$ as $r_0 \to r_s$.

5. Black holes lose energy due to Hawking radiation, whereby one member of a particle-antiparticle pair formed just outside the event horizon crosses the horizon and is claimed by the black hole, allowing the other member (particle or antiparticle) to escape. The result is that an isolated black hole radiates energy at a rate per unit area given by the blackbody law, $f = \sigma T^4$, where f is flux and the temperature is

$$T = \frac{hc^3}{16\pi^2 GMk} \approx 6.2 \times 10^{-8}\ \mathrm{K} \left(\frac{M}{M_\odot}\right)^{-1}.$$

(a) The area of a surface of radius r in the Schwarzschild metric is still $A = 4\pi r^2$, just as in a flat space. Use this expression, with $r = r_s = 2GM/c^2$ to define the area over which the flux is emitted, to write down an expression for the total Hawking luminosity L_H of a black hole, in watts, expressed as a function of $M/M_\odot$. Specifically, verify that $L_H \propto M^{-2}$.

(b) This luminosity carries mass away from the black hole according to

$$\frac{d}{dt}(Mc^2) = -L_H.$$

Substitute your expression for L_H and solve the resulting differential equation for M to show that the *evaporation time* of a black hole of initial mass M, i.e. the time taken for its mass to go to zero, is

$$t_{evap} = \frac{10240\pi^2 G^2 M^3}{hc^4}.$$

(c) Evaluate the evaporation time for a $1\,M_\odot$ black hole.

(d) What mass black hole formed in the Big Bang, 1.3×10^{10} years ago, would just be evaporating today?