PHYS 231: Introductory Astrophysics

Winter 2020

Homework #5 solutions

1. (a) For a neutron star of mass $M_{ns} = 1.3M_\odot$ and radius $R_{ns} = 1.2 \times 10^4$ m, the surface gravity is $GM_{ns}/R_{ns}^2 = 1.2 \times 10^{12}$ m/s^2 $= 1.2 \times 10^{11}$ g.

   (b) The density is $\rho_{ns} = 3M_{ns}/4\pi R_{ns}^3 = 1.1 \times 10^{18}$ kg/m^3.

   (c) The gravitational potential energy is $U = -3GM_{ns}^2/5R_{ns} = -2.2 \times 10^{46}$ J.

   (d) The number density is $n_{ns} = \rho_{ns}/m_n = 6.4 \times 10^{44}$ m^{-3}, so $\ell = n_{ns}^{-1/3} = 1.2 \times 10^{-15}$ m, approximately 1.2 times the neutron radius.

   (e) For rotation speed Ω, the speed at the equator is ΩR. This equals the speed of light when $\Omega = c/R = 2.5 \times 10^4\,\mathrm{rad\,s^{-1}}$, corresponding to a period of $P = 2\pi/\Omega = 0.25$ ms.

2. (a) For a radial gravitational acceleration $a_r = -GM/r^2$, the tidal acceleration on an infalling object of height $\Delta r = H$ is $\Delta a_r = 2GMH/r^3$. For $M = 10\,M_\odot$, $H = 2$ m, and $r = R_s = 2GM/c^2 = 29.5$ km, we find $\Delta a = 2.1 \times 10^8$ m/s^2 $= 2.1 \times 10^7$ g.

   (b) Setting $r = R_s$ in the above expression, the tidal acceleration at the event horizon is

$$\Delta a_r \;=\; \frac{Hc^6}{4G^2M^2}.$$

   For this to be an innocuous $0.001g$, we find $M = (Hc^6/4 \times 10^{-3}G^2g)^{1/2} = 1.5 \times 10^6\,M_\odot$, roughly one third the mass of the black hole at the center of the Milky Way Galaxy.

   (c) The gravitational force on the bottom of the object is to the left, and of magnitude GM/r^2. The force on the top is also of magnitude GM/r^2 to first order, but it is directed at an angle $\theta \approx w/r \ll 1$ (assumed) below the horizontal. Thus the transverse tidal force—the relative vertical force between the bottom and the top—is

$$\Delta a_t \;=\; -\frac{GM}{r^2}\sin\theta \;\approx\; -\frac{GMw}{r^3},$$

   where the negative sign means it tends to compress the object in this direction. Comparing with the radial expression, we see that the transverse and radial tidal accelerations scale the same way with M and r, and are in the (constant) ratio

$$\frac{\Delta a_t}{\Delta a_r} \;=\; -\frac{w}{2\Delta r}.$$

3. (a) The two bodies are orbiting with separation a, each moving in a circular orbit of radius $a/2$ with speed v. Equating accelerations, we have

$$\frac{v^2}{a/2} \;=\; \frac{GM}{a^2},$$

so $v^2 = GM/2a$. The total kinetic energy is $K = Mv^2$ and the potential energy is $U = -GM^2/a$ so the total energy is $E = K + U = -GM^2/2a$. Differentiating, we findv

$$\frac{dE}{dt} = \frac{GM^2}{2a^2}\frac{da}{dt}.$$

(b) The energy is changing due to gravitational wave emission, so

$$\frac{dE}{dt} = \frac{GM^2}{2a^2}\frac{da}{dt} = \dot{E}_{GW} = -\frac{64G^4M^5}{5c^5a^5}.$$

Hence

$$\frac{da}{dt} = -\frac{2a^2}{GM^2}\frac{64G^4M^5}{5c^5a^5} = -\frac{128G^3M^3}{5c^5}a^{-3}.$$

(c) Denoting by A the coefficient of a^{-3} in the above equation, we have

$$\frac{da}{dt} = -Aa^{-3},$$

so if $a = a_0$ at $t = 0$ and $a = 0$ at $t = t_{merge}$, we can say

$$\int_{a_0}^{0} a^3\,da = -A\int_0^{t_{merge}} dt$$

$$t_{merge} = \tfrac{1}{4}a_0^4/A = \frac{5c^5a_0^4}{512G^3M^3}.$$

(d) For $M = M_\odot$ and $a_0 = 0.01$ AU, $t_{merge} = 1.6$ Gyr. For $M = 30M_\odot$ and $a_0 = 0.1$ AU, $t_{merge} = 600$ Myr.

4. (a) The equation of motion of the photon is

$$\frac{dr}{dt} = c\left(1 - \frac{r_s}{r}\right).$$

We can rearrange this to write

$$c\,dt = \frac{dr}{1 - r_s/r} = \frac{r\,dr}{r - r_s}.$$

Integrating from $t = 0, r = r_0$ to t, r, we have

$$[c\,t]_0^t = [r - r_s + r_s\ln(r - r_s)]_{r_0}^r,$$

so

$$ct = (r - r_0) + r_s\ln\left(\frac{r - r_s}{r_0 - r_s}\right).$$

(b) When r is large, far from the black hole, the linear first term dominates the second (logarithmic) term and we have

$$t \approx \frac{r - r_0}{c},$$

as expected in the flat space far from the event horizon.

(c) As $r_0 \to r_s$, $\ln(r_0 - r_s) \to -\infty$, so $t \to \infty$.

5. (a) The total luminosity is $L = 4\pi r^2 \sigma T^4$, where T is given by the expression in the problem and $r = r_s = 2GM/c^2$. Substituting in, we find

$$\begin{aligned} L_{BH} &= 4\pi \left(\frac{2GM}{c^2}\right)^2 \sigma \left(\frac{hc^3}{16\pi^2 GMk}\right)^4 = \frac{1}{4096\pi^7}\left(\frac{\sigma}{G^2M^2}\right)\left(\frac{h^4c^8}{k^4}\right) \\ &= \frac{1}{4096\pi^7}\left(\frac{\sigma}{G^2M^2}\right)\left(\frac{h^4c^8}{k^4}\right) = \frac{hc^2}{30720\pi^2 G^2 M^2}, \end{aligned}$$

where we used the fact that $\sigma = 2\pi^5 k^4/15h^3c^2$ to obtain the final expression. Plugging in the numbers, we have

$$L_{BH} = 9.0 \times 10^{-29}\ \mathrm{W} \left(\frac{M}{M_\odot}\right)^{-2}.$$

(b) We can write

$$\frac{dM}{dt} = -AM^{-2},$$

where

$$A = \frac{hc^4}{30720\pi^2 G^2}.$$

Rewriting the differential equation as

$$M^2\, dM = -A\, dt$$

and integrating from mass M at $t = 0$ to $M = 0$ at $t = t_{evap}$, we find

$$t_{evap} = \frac{M^3}{3A} = \frac{10240\pi^2 G^2 M^3}{hc^4}.$$

(c) For $M = M_\odot$, $t_{evap} = 2.1 \times 10^{67}$ yr.

(d) For $t_{evap} = 1.3 \times 10^{10}$ yr, $M = 8.5 \times 10^{-20} M_\odot = 1.7 \times 10^{11}$ kg.