PHYS 231: Introductory Astrophysics

Winter 2020

Homework #4

(Due: February 19, 2020)

Each problem is worth 20 points.

1. The basic proton-proton fusion reaction in the core of the Sun combines 4 protons to produce 1 helium nucleus, 2 neutrinos (which escape from the Sun), energy in the form of gamma rays, and 2 positrons, which rapidly combine with nearby electrons to produce more gamma rays. The mass of a proton is 1.67262×10^{-27} kg, while the mass of a helium-4 nucleus is 6.64465×10^{-27} kg. As discussed in class, the total local heating from $4p \to {}^4He$ (which ultimately ends up contributing to the luminosity of the Sun) is 26.21 MeV. Given that the the luminosity of the Sun is 3.9×10^{26} W, use this information to calculate (1) the number of reactions occurring per second in the core, and the rates, in Earth masses per year, at which (2) hydrogen is being consumed to form helium, and (3) mass is being converted into energy.

2. Radiation pressure tends to destabilize a star, so we expect that stars in which the radiation pressure $P_{rad} = \frac{1}{3}aT^4$ dominates the thermal gas pressure $P_{gas} = \rho kT/\bar{m}$ should be unstable.

   (a) Use the virial theorem (Maoz Eq. 3.22), with $E_{gr} \approx -GM^2/R$, to show that the mean internal pressure in the star is

   $$P \;\approx\; \left(\frac{4\pi}{81}\right)^{1/3} GM^{2/3}\rho^{4/3},$$

   where $\rho = 3M/4\pi R^3$.

   (b) Show that, if $P_{rad} = P_{gas}$, the point at which we expect the star to become unstable, the total pressure is

   $$P = 2\left(\frac{3}{a}\right)^{1/3}\left(\frac{k\rho}{\bar{m}}\right)^{4/3}.$$

   (c) Equate these two pressures to find the mass of the star at the point of instability. Assume a composition of 71% hydrogen and 29% helium by mass.

3. We can quite easily derive a scaling law for how the nuclear energy generation in a star scales with temperature. The rationale is simple: If a function $f(x)$ scales locally as x^β, then $d\log f/d\log x = x\,d\log f/dx = \beta$, so taking the logarithmic derivative is a convenient way of determining the local power-law scaling.

   Throwing away all the complicated looking constant terms (i.e. terms independent of T), Eq. 3.134 of Maoz takes on a more manageable form:

   $$\epsilon \;\propto\; T^{-2/3}\, e^{-3(E_G/4kT)^{1/3}},$$

where the energy scale E_G is defined in Eq. 3.117.

Take the logarithmic derivative of this expression with respect to T to show that the local scaling is

$$\beta = \left(\frac{E_G}{4kT}\right)^{1/3} - \frac{2}{3}.$$

Evaluate β at $T = 1.5 \times 10^7$ K for the reactions (i) $p + p \to d + e^+ + \nu_e$ and (ii) $p + {}^{12}C \to {}^{13}N + \gamma$.

4. A horizontal branch star is burning helium in its core. It has a surface temperature of 5000 K and is 15 times larger than the Sun.

   (a) What is its luminosity (use the luminosity–temperature–radius relation)?

   (b) The triple-alpha reaction fuses 3 helium-4 nuclei into a carbon-12 nucleus, releasing 7.3 MeV of energy in the process. Calculate its *efficiency*, defined as the total energy released divided by the total mass-energy of the input reactants (in this case $3m_{He}c^2$, where $m_{He} = 6.64 \times 10^{-27}$ kg).

   (c) Assume that the core has a mass of $0.5M_\odot$ and is composed entirely of helium at the start of this stage. How long would it take for all the helium to be consumed, if the luminosity stays constant?

5. (a) Eventually, the envelope of the Sun will drift off as a planetary nebula at a speed of ~ 20 km/s. Suppose an astronomer at Proxima Centauri (distance = 1.3 pc) is watching the sun during its final death throes. How big would the Sun's planetary nebula be (in angular diameter) after 1000 years?

   (b) A white dwarf has mass $0.7\,M_\odot$, radius 10,000 km, and temperature 10^7 K. Estimate its mean density, thermal pressure, surface gravitational acceleration, and electron degeneracy pressure. Compare the degeneracy pressure to the thermal pressure for pure carbon composition.