PHYS 231: Introductory Astrophysics

Winter 2020

Homework #4 solutions

1. The total mass lost in a hydrogen to helium fusion reaction is $\Delta m = 4 \times 1.67262 \times 10^{-27} - 6.64465 \times 10^{-27} = 4.58 \times 10^{-29}$ kg, so the total energy liberated is $\Delta mc^2 = 4.12 \times 10^{-12}$ J $= 25.71$ MeV. The positrons annihilate with electrons to liberate 1.02 MeV $= 1.63 \times 10^{-13}$ J, and 0.52 MeV $= 8.33 \times 10^{-14}$ J is lost in the form of neutrinos, which escape from the Sun. The total contribution to the solar luminosity is thus 26.21 MeV $= 4.20 \times 10^{-12}$ J.

   (1) Since the solar luminosity is $L_\odot = 3.9 \times 10^{26}$ W, in steady state this corresponds to a total of $r = L_\odot/26.21\text{MeV} = 9.3 \times 10^{37}$ fusion reactions per second.

   (2) The total amount of hydrogen consumed per reaction is $\Delta m_H = 4 \times 1.67262 \times 10^{-27}$ kg $= 6.69 \times 10^{-27}$ kg, so hydrogen is consumed at a rate $r\Delta m_H = 6.2 \times 10^{11}$ kg/s $= 3.3 \times 10^{-6}\ M_\oplus$/yr.

   (3) The total amount of mass consumed per reaction is $\Delta m = 4.58 \times 10^{-29}$ kg (0.7% of the hydrogen mass consumed), so the mass consumption rate is $r\Delta m = 4.3 \times 10^9$ kg/s $= 2.2 \times 10^{-8}\ M_\oplus$/yr.

2. (a) Maoz Eq. 3.22 says

$$P = -\frac{E_{gr}}{3V} \approx \frac{GM^2}{3RV}.$$

   Writing $V = M/\rho$ and $R = (3M/4\pi\rho)^{1/3}$, this implies

$$P \approx \frac{GM\rho}{3}\left(\frac{4\pi\rho}{3M}\right)^{1/3} = \left(\frac{4\pi}{81}\right)^{1/3} GM^{2/3}\rho^{4/3}.$$

   (b) If $P_{rad} = P_{gas}$, then $\frac{1}{3}aT^4 = \rho k/\bar{m}$, so $T^3 = 3\rho kT/a\bar{m}$, and

$$P = P_{rad} + P_{gas} = 2P_{rad} = \tfrac{2}{3}aT^4 = \tfrac{2}{3}a\left(\frac{3\rho k}{a\bar{m}}\right)^{4/3} = 2\left(\frac{3}{a}\right)^{1/3}\left(\frac{k\rho}{\bar{m}}\right)^{4/3}.$$

   (c) Setting these two expressions equal (and noting that the $\rho^{4/3}$ factors cancel), we have

$$\left(\frac{4\pi}{81}\right)^{1/3} GM^{2/3} = 2\left(\frac{3}{a}\right)^{1/3}\left(\frac{k}{\bar{m}}\right)^{4/3}$$

   so

$$M = 9\left(\frac{6}{\pi}\right)^{1/2} G^{-3/2}\, a^{-1/2}\left(\frac{k}{\bar{m}}\right)^2.$$

   For the given composition, $\bar{m} = 0.61\, m_p$. Plugging in the numbers, we find a maximum mass of $M \approx 76\, M_\odot$.

3. Given that

$$\epsilon \;=\; A\,T^{-2/3}\,e^{-3(E_G/4kT)^{1/3}},$$

where A is a constant, we have

$$\log\epsilon \;=\; \log A - \tfrac{2}{3}\log T - 3\left(\frac{E_G}{4kT}\right)^{1/3}.$$

Taking the logarithmic derivative, we find

$$\beta \;=\; T\frac{d\log\epsilon}{dT} \;=\; -\frac{2}{3} + 3\left(\frac{E_G}{4k}\right)^{1/3}\frac{1}{3}T^{-1/3} \;=\; \left(\frac{E_G}{4kT}\right)^{1/3} - \frac{2}{3}.$$

The Gamow energy is

$$E_G = 493\,\mathrm{keV}\,(Z_A Z_B)^2\left(\frac{\mu}{\frac{1}{2}m_p}\right),$$

where Z_A and Z_B are the atomic numbers (charges) of the interacting nuclei and $\mu = m_A m_B/(m_A+m_B)$.

For $T = 1.5\times10^7$ K, $kT = 1.29$ keV, so (i) for the $p-p$ reaction we have $Z_A = Z_B = 1, \mu = \frac{1}{2}m_p$, so $E_G = 493$ keV and $\beta = 3.9$; while (ii) for the $p-C$ reaction we have $Z_A = 1, Z_B = 6, \mu = \frac{12}{13}m_p$, so $E_G = 32.8$ MeV and $\beta = 17.8$.

4. (a) The luminosity $L \propto R^2T^4$, so

$$L \;=\; \left(\frac{R}{R_\odot}\right)^2\left(\frac{T}{T_\odot}\right)^4 L_\odot \;=\; 15^2\left(\frac{5000}{5800}\right)^4 L_\odot \;=\; 124\,L_\odot.$$

(b) The efficiency is $\epsilon = 7.3$ MeV / $3m_{He}c^2 = 0.00065$.

(c) Every 7.3 MeV of energy produced uses up $3m_{He} = 1.99\times10^{-26}$ kg of helium, so the helium consumption rate is $1.99\times10^{-26}L/7.3$ MeV kg/s $= 8.26\times10^{14}$ kg/s. At this rate, the entire core will be consumed in 38 million years.

5. (a) After 1000 years the planetary nebula has a diameter of 2×20 km/s $\times$ 1000 years $= 1.26\times10^{12}$ km $= 0.041$ pc. Seen from a distance of 1.3 pc, this corresponds to an angular diameter of $0.041/1.3 = 0.032$ radians, or 1.8 degrees.

(b) For mass $M = 0.7\,M_\odot$ and radius $R = 10^4$ km, the mean density is $\bar{\rho} = 3M/4\pi R^3 = 3.3\times10^8\,\mathrm{kg/m^3}$. The surface gravity is $GM/R^2 = 9.3\times10^5\,\mathrm{m/s^2} = 9.5\times10^4 g$. For pure carbon composition, the formula in the book (which neglects the carbon nuclei) gives $\bar{m} = 2m_p$. Alternatively, including the carbon nuclei, we find $n = 7\rho/(12m_p)$, so $\bar{m} = \rho/n = 1.71m_p$. The thermal pressure then is $P = (\rho/\bar{m})kT = 1.4\times10^{19}$ Pa (or 1.6×10^{19} Pa) for $T = 10^7$ K. The electron degeneracy pressure (for $Z/A = 0.5$) is $P_e = 5.0\times10^{20}$ Pa, some 30 times larger than the thermal pressure.