PHYS 231: Introductory Astrophysics

Winter 2020

Homework #3 solutions

1. If X, Y, and $Z = 1 - X - Y$ are the fractions by mass of hydrogen, helium, and metals, then, as discussed in class, in a gas of density ρ, the nucleon number density n_n, the electron number density n_e, and the mean mass per particle $\bar{m}$ are

$$\begin{aligned}
n_n &= \frac{X\rho}{m_H} + \frac{Y\rho}{4m_H} + \frac{Z\rho}{Am_H} \\
&= \frac{\rho}{m_H}\left(X + \tfrac{1}{4}Y\right) \\
n_e &= \frac{X\rho}{m_H}\cdot 1 + \frac{Y\rho}{4m_H}\cdot 2 + \frac{Z\rho}{Am_H}\cdot\frac{A}{2} \\
&= \frac{\rho}{m_H}\left(X + \tfrac{1}{2}Y + \tfrac{1}{2}Z\right) \\
&= \frac{\rho}{2m_H}(1+X) \\
\bar{m} &= \frac{\rho}{n_n + n_e} \\
&= \frac{2m_H}{1 + 3X + \frac{1}{2}Y},
\end{aligned}$$

where the mean molecular weight per metal A is large and we therefore ignore the contribution of metals to n_n. Hence the solutions for the three environments are

$$\begin{aligned}
&\text{(i)}\ \ n_e = 5.24\times10^{27}\ \mathrm{m}^{-3},\ \ell = 2.85\ \mathrm{m},\ \bar{m} = 9.91\times10^{-28}\ \mathrm{kg} = 0.593\ m_H, \\
&\text{(ii)}\ \ n_e = 5.12\times10^{29}\ \mathrm{m}^{-3},\ \ell = 2.92\ \mathrm{cm},\ \bar{m} = 1.02\times10^{-27}\ \mathrm{kg} = 0.613\ m_H, \\
&\text{(iii)}\ \ n_e = 4.01\times10^{31}\ \mathrm{m}^{-3},\ \ell = 0.372\ \mathrm{mm},\ \bar{m} = 1.43\times10^{-27}\ \mathrm{kg} = 0.855\ m_H.
\end{aligned}$$

2. The intensity varies according to

$$\frac{dI}{dx} = -\alpha I + \epsilon.$$

(a) Writing $y = I - \epsilon/\alpha$, where ϵ/α a constant, it is clear that

$$\frac{dy}{dx} = \frac{dI}{dx},$$

so

$$\frac{dy}{dx} = -\alpha\left(I - \frac{\epsilon}{\alpha}\right) = -\alpha y.$$

(b) The solution to this equation is

$$y = y_0\, e^{-\alpha x},$$

or

$$
\begin{aligned}
I - \frac{\epsilon}{\alpha} &= \left(I_0 - \frac{\epsilon}{\alpha}\right) e^{-\alpha x} \\
I &= I_0\, e^{-\alpha x} + \frac{\epsilon}{\alpha}\left(1 - e^{-\alpha x}\right).
\end{aligned}
$$

Hence the initial value I_0 is effectively forgotten for $\alpha x \gg 1$, or $x \gg 1/\alpha$, and the solution for x in that regime is $I = \epsilon/\alpha$.

3. Equation (3.22) of Maoz is obtained by integrating $4\pi r^3$ times the equation of hydrostatic equilibrium over the entire star:

$$V\bar{P} = -\tfrac{1}{3}E_{gr},$$

where $\bar{P} = \int_0^{R_*} 4\pi r^2 P(r)dr/V$ is the volume-averaged pressure, V is the volume of the star, and E_{gr} is the total gravitational potential energy. For a non-relativistic gas, $P = \frac{2}{3}u_{th}$, where u_{th} is the mean thermal energy density, so $V\bar{P} = \frac{2}{3}E_{th}$, where E_{th} is the star's total thermal energy, and we find $E_{th} = -\frac{1}{2}E_{gr}$. Now suppose the gas is relativistic, with equation of state $P = \frac{1}{3}u_{th}$. The same reasoning now implies $E_{th} = -E_{gr}$.

Thus, for a relativistic gas, the total energy $E^{tot} = E_{th} + E_{gr} = 0$, and the star is only marginally bound. Very massive, hot stars are close to being unstable.

4. The ratio of pressures is

$$p \equiv \frac{P_{rad}}{P_{gas}} = \frac{\bar{m}aT^3}{3\rho k},$$

where $\bar{m}$ is as in Problem 1. Thus we have $p =$ (i) 1.81×10^{-6}, (ii) 1.87×10^{-5}, and (iii) 2.61×10^{-4}.