PHYS 305: Computational Physics II

Winter 2022

Homework #3

(Due: February 8, 2022)

Each problem is worth 10 points. Upload your solutions to Learn with a title including PHYS 305 and the Homework number. The PDF upload should contain all discussion, results, and graphs requested, and files containing Python scripts for all programs written.

1. The general form of a symplectic scheme to solve the conservative second-order dynamical system

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= a(x) \end{aligned}$$

is

$$\begin{aligned} v &= v + d_0\, a(x)\, \delta t \\ x &= x + c_0\, v\, \delta t \\ &\cdot \\ &\cdot \\ &\cdot \\ v &= v + d_{s-1}\, a(x)\, \delta t \\ x &= x + c_{s-1}\, v\, \delta t \end{aligned}$$

This generalized "kick–drift" method is defined by s, the number of stages, and the coefficients c_i and d_i.

(a) Implement the following scheme, defined by $s = 4$, $c = (1/r, q/r, q/r, 1/r)$, $d = (0, 2/r, -2p/r, 2/r)$, where $p = 2^{1/3}$, $q = 1 - p$, $r = 4 - 2p$.

(b) Apply this integrator to the Duffing oscillator with parameters $\alpha = -2$, $\beta = 1$, $\delta = 0$ (as used in class), with initial conditions $x = 1$ and $v = 1.5$ at $t = 0$. Integrate the system from $t = 0$ to $t = 20$ with step size $\delta t = 0.01$ and plot (i) the phase portrait (v versus x) and (ii) the energy error $E(t) - E(0)$ as a function of t.

(c) For the same parameters and initial conditions, make a log–log plot of the absolute value of the final energy error $|E(20) - E(0)|$ as a function of δt for $\delta t = 2^{-n}, n = 1, \ldots 13$, and hence determine the order of this scheme.

(d) Run the calculation from $t = 0$ to $t = 20$ with $\delta t = 0.01$, and then backwards from $t = 20$ to $t = 0$ with the same δt, and print the values of x and v at the end. Is the scheme reversible?

2. (a) Do Exercise 5.2 on the course web page. Specifically, carry out three integrations of the Sun–Earth–Jupiter system, varying Jupiter's mass M_J and orbital semi-major axis R_J as follows: (i) $M_J = 0.01, R_J = 3.0$, (ii) $M_J = 0.02, R_J = 2.1$, (iii) $M_J = 0.03, R_J = 2.0$. Take $\epsilon = 0$ and run to $t = 1000$. In each case, plot Earth's orbital eccentricity e as a function of time and determine the maximum eccentricity e_{max} reached over the course of the calculation.

   (b) Repeat the orbital calculations (without the plots!) in part (a) for a grid of initial conditions in M_J and R_J: $M_J = 0.01, \ldots, 0.2$ in steps of 0.01, and $R_J = 1.2, \ldots, 2.5$ in steps of 0.1. Take $\epsilon = 0$ and run to $t = 2000$. For definiteness, let's call an orbit for which $e_{max} > 0.5$ an unstable outcome. Plot the outcome of each run as a small circle on the $M_J - R_J$ plane, indicating stability or instability by color, and using the size of the circle to indicate the eccentricity of a stable orbit.

   Note: this may take half an hour or more to run. Do the calculation first with a coarser grid in M_J and R_J to get the graphics right!

   If your computer can stand it, do each (M_J, R_J) run four times, choosing the initial phase of Jupiter's orbit to be $0, \pi/2, \pi$, and $3\pi/2$, and average the resulting e_{max} values.

   (c) What can you conclude about the region of stability in this diagram?

3. (a) Set up a three-body system in the following "Pythagorean" configuration:

```
particle 0:  mass = 5, pos = [0, 0, 0], vel = [0, 0, 0]
particle 1:  mass = 4, pos = [3, 0.5, 0], vel = [0, 0, 0]
particle 2:  mass = 3, pos = [0, 4.5, 0], vel = [0, 0, 0]
```

   Transform to the center of mass frame before starting the integration (for visualization purposes). Use $\epsilon = 0.1$ and integrate the system using the second-order kick-drift-kick scheme with $\delta t = 0.001$, and run to time $t = 50$. Plot (1) the trajectories of all three bodies in the $x - y$ plane, and (2) the x-coordinates of all three bodies as functions of time.

   (b) Repeat part (a) for (i) $\delta t = 0.0005$ and (ii) $\delta t = 0.001$, but using the symplectic integrator you developed for problem 1.

   (c) Now simultaneously run two versions of the system, one with the same initial conditions as in part (a), the other with the initial y-velocity of particle 2 equal to 0.001. Plot the rms "distance" D between the two simulations, defined as

$$D = \sqrt{\frac{1}{M} \sum_{i=0}^{2} m_i \, |\mathbf{x}_i - \mathbf{x}'_i|^2},$$

   as a function of time, where $\mathbf{x}$ refers to the original system, $\mathbf{x}'$ to the modified one, and M is the total mass. Can you interpret this plot?