PHYS 305: Computational Physics II

Winter 2022

Homework #2

(Due: January 25, 2022)

Each problem is worth 10 points. Upload your solutions to Learn with a title including PHYS 305 and the Homework number. The PDF upload should contain all discussion, results, and graphs requested, and files containing Python scripts for all programs written.

1. The general form of a Runge-Kutta scheme (state $n \to n+1$) to solve the ODE

$$\frac{dy}{dx} = f(x, y)$$

is

$$\begin{aligned}
\delta y_0 &= \delta x\, f(x_n,\, y_n) \\
\delta y_1 &= \delta x\, f(x_n + a_1\delta x,\, y_n + b_{10}\delta y_0) \\
\delta y_2 &= \delta x\, f(x_n + a_2\delta x,\, y_n + b_{20}\delta y_0 + b_{21}\delta y_1) \\
&\ \ \vdots \\
\delta y_{s-1} &= \delta x\, f\left(x_n + a_{s-1}\delta x,\, y_n + \sum_{i=0}^{s-2} b_{s-1,i}\,\delta y_i\right) \\
y_{n+1} &= y_n + \sum_{i=0}^{s-1} c_i \delta y_i
\end{aligned}$$

(see Numerical Recipes, Sec. 17.2). This is not quite the syntax used previously in class or in the book—the count starts at 0 to facilitate translation of the algorithm into Python. The method is defined by s, the number of stages, the offsets a_i and b_{ji}, and the weights c_i.

(a) Implement the following scheme, defined by $s = 6$, $a = (0, 1/5, 3/10, 3/5, 1, 7/8)$, $c = (37/378, 0, 250/621, 125/594, 0, 512/1771)$, and

$$b = \begin{pmatrix}
0 & & & & & \\
1/5 & 0 & & & & \\
3/40 & 9/40 & 0 & & & \\
3/10 & -9/10 & 6/5 & 0 & & \\
-11/54 & 5/2 & -70/27 & 35/27 & 0 & \\
1631/55296 & 175/512 & 575/13824 & 44275/110592 & 253/4096 & 0
\end{pmatrix},$$

where all elements on and above the diagonal are zero.

(b) Apply this integrator to the Duffing oscillator with parameters $\alpha = -2$, $\beta = 1$, $\delta = 0$ (as used in class), with initial conditions $y = 1$ and $y' = 1.5$ at $x = 0$. Integrate the system from

$x = 0$ to $x = 20$ with step size $\delta x = 0.01$ and plot (i) the phase portrait (y versus x) and (ii) the energy error $E(x) - E(0)$ as a function of x.

(c) For the same parameters and initial conditions, make a log–log plot of the absolute value of the final energy error $|E(20) - E(0)|$ as a function of δx for $\delta x = 2^{-n}, n = 1, \ldots 13$, and hence determine the order of this scheme.

(d) Run the calculation from $x = 0$ to $x = 20$ with $\delta x = 0.01$, and then backwards from $x = 20$ to $x = 0$ with the same δx, and print the values of y and y' at the end. Is the scheme reversible?

2. Apply the integrator from problem 1 (or RK4, if you had problems) to the chaotic oscillator problem discussed in class. You may use the code `ddd.py` as a starting point. We will explore how the solution to the problem changes as first damping and then driving are introduced. Choose $\alpha = -1, \beta = 1, \gamma = 0$ and take as initial conditions $y[0] = 1.5, y[1] = 0$ at $x = 0$.

   (a) Modify the script to show two graphs in the same frame: the time sequence ($y[0]$ versus x) at the left and the phase portrait ($y[1]$ versus $y[0]$) at right. Plot the results for $\delta = 0$ and $\delta = 0.3$.

   (b) Now, with $\delta = 0.3$, set $\omega = 1.2$ and $\gamma = 0.2$ and graph the motion. Note how the initial oscillation dies away and the system ends up oscillating at the driving frequency ω.

   (c) From here on, we will discard the decay of the initial conditions by only plotting results after some time. Modify the program to start plotting at $x = 200$ and continue until $x = 500$. Plot the results for $\gamma = 0.28, 0.29, 0.37, 0.5$, and 0.65. You should see period doubling and a transition into and out of chaotic motion.

   (d) Now compute so-called *Poincaré sections* of the data, as follows. Instead of plotting every data point in the phase portrait, plot only the points where x is an integral multiple of the driving period $P = 2\pi/\omega$. In other words, when the previous value of x was less than nP and the current value of x is greater than nP, for some n, use linear interpolation to determine the values of $y[0]$ and $y[1]$ at $x = nP$. Plot both the phase portrait and the Poincaré section for each of the γ values studied in part (c).

   (e) Compare the results obtained with the high-order integrator to those using the Midpoint method. How does the behavior you see depend on the choice of integration scheme?