### Vectors and Linear Transformations

A vector space $V$ is a set of things called **basis vectors** and some rules for making linear combinations of them:

- $ax + by$ is a vector if $x$, $y$ are vectors and $a, b$ are numbers.

A linear transformation $L$ is a map from one vector space to another that obeys the superposition principle:

$$L(ax + by) = aLx + bLy$$

Every linear transformation can be represented by a matrix acting on a column vector and vice versa. This is important.

An inner product $\langle x | y \rangle$ maps two vectors to a number. The usual example is $x_1^* y_1 + x_2^* y_2 + \cdots$ but others exist. The inner product of a vector with itself defines a norm.

### Matrix Arithmetic

To multiply two matrices $AB$, do this:  
$$[AB]_{ij} = \sum_k A_{ik} B_{kj}$$
(Note: a column vector is just a $n \times 1$ matrix.)

$(AB)x$ produces the same vector as “do $B$, then do $A$ to $x$.”

Matrices add component-wise, and $(A + B)x = Ax + Bx$.

To transpose $M$, swap its rows and columns: $[M^T]_{ij} = M_{ji}$
An (anti) symmetric matrix equals its (minus) transpose.

The adjoint of $M$ is its conjugate transpose: $[M^\dagger]_{ij} = M_{ji}^\ast$.
Adjoints obey the rule $\langle x | M y \rangle = \langle M^\dagger x | y \rangle$.

The inverse $M^{-1}$ has determinant $(\det[M])^{-1}$ if $\det[M] \neq 0$.
A singular matrix has determinant 0 and can't be inverted.

Transposes, adjoints and inverses obey a “backwards” rule:

$$(AB)^{-1} = B^{-1} A^{-1} \quad (AB)^\dagger = B^\dagger A^\dagger \quad (AB)^\ast = B^\ast A^\ast$$

### Unitary / Orthogonal

Unitary matrices obey $U^{-1} = U^\dagger$. Real unitary matrices are orthogonal. $U$ matrices preserve the usual inner product: $\langle Ux | Uy \rangle = \langle x | y \rangle$. Each eigenvalue of $U$ and the determinant of $U$ must have complex magnitude $1$.

The columns of $U$ form an orthonormal basis for $V$ (and so do the rows) if and only if $U$ is unitary. Two matrices $L$ and $M$ are similar if $M = ULU^{-1}$ for some unitary $U$.

Every rotation and/or parity transformation between two orthonormal bases is represented by a $U$ and vice versa.

### Hermitian / Symmetric

Hermitian matrices are self-adjoint: $H^\dagger = H$. Real symmetric square matrices are Hermitian.

Eigenvalues of $H$ are real (but might be degenerate!).
Eigenvectors of $H$ form an orthogonal basis for $V$.
(Eigenvectors corresponding to the same eigenvalue are not unique, but it is always possible to choose orthogonal ones.)

A real linear combination of Hermitian matrices is Hermitian.

### Eigensystems and the Spectral Theorem

A normal matrix $N$ satisfies $NN^\dagger = N^\dagger N$. Every normal matrix is similar to a diagonal matrix: $N = UDU^{-1}$ where $D$ is diagonal. Elements of $D$ are eigenvalues and columns of $U$ are eigenvectors of $N$. If $N$ is Hermitian, then $U$ is unitary. $\lambda_i$ is an eigenvector of $N$ with eigenvalue $\lambda_i$ if and only if $N v_j = \lambda_j v_j$. The (complex) phase of an eigenvector is arbitrary.

The spectrum of $N$ (the set of its eigenvalues) can be found by solving $\det[N - \lambda I] = 0$, the characteristic polynomial of $N$. The product of all eigenvalues of $N$ is $\det[N]$ and the sum of eigenvalues is $\text{tr}[N]$, the trace of $N$ (the sum of its diagonal elements). Two similar matrices $L$ and $M$ have the same spectrum, determinant, and trace (but the converse is not true).

### Misc. Terminology

A matrix $P$ is idempotent if $PP = P$. An idempotent Hermitian matrix is a projection. A positive-definite matrix has only positive real eigenvalues. $Z$ is nilpotent if $Z^n = 0$ for some number $n$. The commutator of $L$ and $M$ is $[L,M] = LM - ML$.

### Matrix Exponentials

The exponential map of a matrix $M$ is $\exp[M] = 1 + M + \frac{1}{2!} M^2 + \cdots + \frac{1}{k!} M^k + \cdots$. The solution to the differential equation $\frac{\text{d}}{\text{d}t} x(t) = M x(t)$ is $x(t) = \exp[M t] \cdot x(0)$. EXP has some, but not all, of the properties of the function $e^x$:

- in general: 
  $$(e^M)^{-1} = e^{-M} \quad (e^M)^T = e^{M^T} \quad (e^M)^\dagger = e^{M^\dagger}$$
  $e^{(a + b)M} = e^{aM} e^{bM}$
  $\det[e^M] = e^{\text{tr}[M]}$
- only if $M$ and $N$ commute: $e^{M+N} = e^M e^N \quad e^{N} M e^{-N} = M$ only if $N$ is invertible: $e^{N M N^{-1}} = N e^N N^{-1}$