PHYS 631: General Relativity

Homework #4

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1. **(Schutz 6.29)** In polar coordinates, calculate the Riemann curvature tensor of the sphere of unit radius whose metric is $g_{\theta\theta} = r^2, g_{\phi\phi} = r^2 \sin^2\theta, g_{\theta\phi} = 0$.
Solution:
The metric for polar coordinate on the surface of unit sphere is

$$\begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$$

The christoffel symbols are given by

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma} \right)$$

The only non zero derivative of metric is with respect to θ so we get

$$\Gamma^{\theta}{}_{\phi\phi} = \frac{1}{2} g^{\theta\theta} \left(-g_{\phi\phi,\theta} \right) = -\frac{1}{2} \sin 2\theta$$

Similarly the other non zero Christoffel symbols are

$$\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \frac{\cos\theta}{\sin\theta}$$

And the Riemann tensor is given by

$$\begin{aligned} R^{\alpha}{}_{\beta\mu\nu} &= \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\beta\nu,\mu} \\ R_{\alpha\beta\mu\nu} &= g_{\alpha\lambda} \left(\Gamma^{\lambda}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\lambda}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu} - \Gamma^{\lambda}_{\beta\mu,\nu} + \Gamma^{\lambda}_{\beta\nu,\mu} \right) \end{aligned}$$

Calculating

$$\begin{aligned} R_{\phi\theta\phi\theta} &= g_{\phi\phi} \left(\Gamma^{\phi}_{\sigma\phi}\Gamma^{\sigma}_{\theta\theta} - \Gamma^{\phi}_{\sigma\theta}\Gamma^{\sigma}_{\theta\phi} - \Gamma^{\phi}_{\theta\phi,\theta} + \Gamma^{\phi}_{\theta\theta,\phi} \right) \\ &= \sin^2\theta \left(\Gamma^{\phi}_{\sigma\phi}\Gamma^{\sigma}_{\theta\theta} - \Gamma^{\phi}_{\sigma\theta}\Gamma^{\sigma}_{\theta\phi} - \Gamma^{\phi}_{\theta\phi,\theta} + \Gamma^{\phi}_{\theta\theta,\phi} \right) \\ &= \sin^2\theta \left(-\Gamma^{\phi}_{\phi\theta}\Gamma^{\phi}_{\theta\phi} - \Gamma^{\phi}_{\theta\phi,\theta} \right) \\ &= \sin^2\theta \left(-\frac{\cos^2\theta}{\sin^2\theta} + \frac{1}{\sin^2\theta} \right) \\ &= \sin^2\theta \end{aligned}$$

Now we can permute the coordinate with the symmetry property to obtain

$$R_{\phi\theta\theta\phi} = -\sin^2\theta \qquad R_{\theta\phi\phi\theta} = -\sin^2\theta \qquad R_{\theta\phi\theta\phi} = \sin^2\theta$$

These are the non zero components of Riemann tensor. □

2. **(Schutz 6.30)** Calculate the Riemann curvature tensor of the cylinder.
Solution:
The line element ins the cylindrical coordinate system is

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2$$

So the metric in is

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Christoffel symbols are given by

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma} \right)$$

The only non zero derivative of metric is with respect to θ so we get

$$\Gamma^{r}{}_{\phi\phi} = \frac{1}{2} g^{rr} \left(-g_{\phi\phi,r} \right) = -r$$

Similarly the other non zero Christoffel symbols are

$$\Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{1}{r}$$

And the Riemann tensor is given by

$$\begin{aligned} R^{\alpha}{}_{\beta\mu\nu} &= \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\beta\nu,\mu} \\ R_{\alpha\beta\mu\nu} &= g_{\alpha\lambda} \left(\Gamma^{\lambda}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\lambda}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu} - \Gamma^{\lambda}_{\beta\mu,\nu} + \Gamma^{\lambda}_{\beta\nu,\mu} \right) \end{aligned}$$

Calculating

$$\begin{aligned} R_{\phi r\phi r} &= g_{\phi\phi} \left(\Gamma^{\phi}_{\sigma\phi}\Gamma^{\sigma}_{rr} - \Gamma^{\phi}_{\sigma r}\Gamma^{\sigma}_{r\phi} - \Gamma^{\phi}_{r\phi,r} + \Gamma^{\phi}_{rr,\phi} \right) \\ &= r^2 \left(\Gamma^{\phi}_{\sigma\phi}\Gamma^{\sigma}_{rr} - \Gamma^{\phi}_{\sigma r}\Gamma^{\sigma}_{r\phi} - \Gamma^{\phi}_{r\phi,r} + \Gamma^{\phi}_{rr,\phi} \right) \\ &= r^2 \left(-\Gamma^{\phi}_{\phi r}\Gamma^{\phi}_{r\phi} - \Gamma^{\phi}_{r\phi,r} \right) \\ &= r^2 \left(-\frac{1}{r^2} + \frac{1}{r^2} \right) \\ &= 0 \end{aligned}$$

Now we can permute the coordinates and with symmetry all the rest are zero too.

$$R_{\phi rr\phi} = 0 \qquad R_{r\phi\phi r} = 0 \qquad R_{r\phi r\phi} = 0$$

So all the components of Riemann tensor are zero, showing that the surface of cylinder is a flat surface. □

3. One way of describing the metric of a flat, homogeneous, expanding universe is:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2 & 0 \\ 0 & 0 & 0 & a(t)^2 \end{pmatrix}$$

where $a(t)$ is a function of time only, and the coordinates are

$$x^{\mu} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

(a) Compute all non vanishing terms of the Riemann Tensor.
Solution:

The Christoffel symbols are given by

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma} \right)$$

The only non zero derivative of metric is with respect to t so we get

$$\Gamma^{t}{}_{xx} = \frac{1}{2} g^{tt} \left(-g_{xx,t} \right) = a\dot{a}$$

These are true for y and z coordinates.

$$\Gamma^{t}{}_{yy} = \frac{1}{2} g^{tt} \left(-g_{yy,t} \right) = a\dot{a} \qquad \Gamma^{t}{}_{zz} = \frac{1}{2} g^{tt} \left(-g_{zz,t} \right) = a\dot{a}$$

Similarly the other non zero Christoffel symbols are

$$\Gamma^{x}{}_{tx} = \Gamma^{x}{}_{xt} = \frac{\dot{a}}{a}$$

These are also true for y and z.

$$\Gamma^{y}{}_{ty} = \Gamma^{y}{}_{yt} = \frac{\dot{a}}{a} \qquad \Gamma^{z}{}_{tz} = \Gamma^{z}{}_{zt} = \frac{\dot{a}}{a}$$

And the Riemann tensor is given by

$$\begin{aligned} R^{\alpha}{}_{\beta\mu\nu} &= \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\beta\nu,\mu} \\ R_{\alpha\beta\mu\nu} &= g_{\alpha\lambda} \left(\Gamma^{\lambda}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\lambda}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu} - \Gamma^{\lambda}_{\beta\mu,\nu} + \Gamma^{\lambda}_{\beta\nu,\mu} \right) \end{aligned}$$

Calculating

$$\begin{aligned} R_{xtxt} &= g_{xx} \left(\Gamma^{x}_{\sigma x}\Gamma^{\sigma}_{tt} - \Gamma^{x}_{\sigma t}\Gamma^{\sigma}_{tx} - \Gamma^{x}_{tx,t} + \Gamma^{x}_{tt,x} \right) \\ &= a^2 \left(\Gamma^{x}_{\sigma x}\Gamma^{\sigma}_{tt} - \Gamma^{x}_{\sigma t}\Gamma^{\sigma}_{tx} - \Gamma^{x}_{tx,t} + \Gamma^{x}_{tt,x} \right) \\ &= a^2 \left(-\Gamma^{x}_{xt}\Gamma^{x}_{tx} - \Gamma^{x}_{tx,t} \right) \\ &= a^2 \left(-\frac{\dot{a}^2}{a^2} + \frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} \right) \\ &= a\ddot{a} \end{aligned}$$

Now we can permute the coordinate with the symmetry property to obtain

$$R_{xttx} = -a\ddot{a} \qquad R_{txxt} = -a\ddot{a} \qquad R_{txtx} = a\ddot{a}$$

Similarly the rest of the values can be calculated as

$$R_{yxxy} = -a^2\dot{a}^2$$

The rest of them can be obtained by permuting the index using the (anti-)symmetry property.

$$R_{zxxz} = R_{zyyz} = R_{yzzy} = R_{xzzx} = R_{xyyx} = -a^2\dot{a}$$

□

(b) Compute all Non-vanishing terms of the Ricci Tensor.
Solution:
The raised version of Riemann tensor is

$$R^{\alpha}{}_{\beta\gamma\mu} = g^{\alpha\sigma} R_{\sigma\beta\gamma\mu}$$

The first index non vanishing term is

$$\begin{aligned} R^{x}_{ttx} &= g^{tt}\cancelto{0}{R_{tttx}} + g^{xx} R_{xttx} + g^{yy}\cancelto{0}{R_{yttx}} + g^{zz}\cancelto{0}{R_{zttx}} \\ &= a^{-2}\left(-a\ddot{a}\right) = -\frac{\ddot{a}}{a} \end{aligned}$$

Using the symmetry property and the elements of metric we get the rest of components of Riemann tensor as

$$\begin{aligned} R^{x}_{ttx} &= -\frac{\ddot{a}}{a} \qquad & R^{x}_{txt} &= \frac{\ddot{a}}{a} \\ R^{y}_{tty} &= -\frac{\ddot{a}}{a} \qquad & R^{y}_{tyt} &= \frac{\ddot{a}}{a} \\ R^{z}_{ttz} &= -\frac{\ddot{a}}{a} \qquad & R^{z}_{tzt} &= \frac{\ddot{a}}{a} \end{aligned}$$

Now the components of Ricci tensor in terms of elements of Riemann Rensor are

$$R_{\alpha\beta} = g^{\mu\nu} R^{\nu}_{\alpha\mu\beta}$$

Specifically for R_{tt} we get

$$\begin{aligned} R_{tt} &= g^{tt}\cancelto{0}{R^{t}_{ttt}} + g^{xx} R^{x}_{txt} + g^{yy} R^{y}_{tyt} + g^{zz} R^{z}_{tzt} \\ &= -a^{-2}\ddot{a}a - a^{-2}\ddot{a}a - a^{-2}\ddot{a}a \\ &= -3\ddot{a}/a \end{aligned}$$

Similarly rest of the components can be calculated. They are

$$R_{xx} = R_{yy} = R_{zz} = a\ddot{a} + 2\dot{a}^2$$

These are the components of Ricci tensor □

(c) Compute Einstein Tensor.
Solution:
The components of Einstein tensor are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{1}$$

The Ricci scalar can be calculated by contracting the Ricci tensor as

$$R = R^{t}_{t} + R^{x}_{x} + R^{y}_{y} + R^{z}_{z} = 6\frac{a\ddot{a} + \dot{a}^2}{a^2} \tag{2}$$

Now the Einstein tensor simply is the substitution (2) into the (1). The first component of this tensor is

$$G_{tt} = R_{tt} - \frac{1}{2} g_{tt} R = -3\frac{\ddot{a}}{a} + \frac{1}{2}\frac{6\left(a\ddot{a} + \dot{a}^2\right)}{a^2} = 3\frac{\dot{a}^2}{a^2}$$

Similarly the rest of the components can be calculated.

$$G_{xx} = R_{xx} - \frac{1}{2} g_{xx} R = a\ddot{a} + 2\dot{a}^2 - \frac{1}{2} a^2 \cdot \frac{6(a\ddot{a} + \dot{a}^2)}{a^2} = -2a\ddot{a} - \dot{a}^2$$

$$G_{xx} = G_{yy} = G_{zz} = -2a\ddot{a} - \dot{a}^2$$

The raised version of Einstein tensor similarly are[1].

$$G^{tt} = \frac{3}{2}\frac{\dot{a}^2}{a^2} \qquad G^{xx} = G^{yy} = G^{zz} = -\frac{\dot{a}^2 + 2a\ddot{a}}{a^4}$$

These are the required components of Einstein tensor. □

4. **(Schutz 6.35)** Compute 20 independent components of $R_{\alpha\beta\mu\nu}$for a manifold with line element $ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$, where Φ and Λ are arbitrary functions fo the coordinate r alone.
Solution:
Writing down the metric from the given expression for line element

$$g_{tt} = -e^{2\Phi}; \qquad g_{rr} = e^{2\Lambda}; \qquad g_{\theta\theta} = r^2; \qquad g_{\phi\phi} = r^2\sin^2\theta$$

The inverse metric is

$$g^{tt} = -e^{-2\Phi}; \qquad g^{rr} = e^{-2\Lambda}; \qquad g^{\theta\theta} = \frac{1}{r^2}; \qquad g^{\phi\phi} = \frac{1}{r^2\sin^2\theta}$$

The Christoffel symbols can be calculated by the expression

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma}\right)$$

Evaluating the these we get

$$\begin{aligned}
&\Gamma^t_{rt} = \Gamma^t_{tr} = \Phi_{,r} \\
&\Gamma^r_{\phi\phi} = -re^{-2\Lambda}\sin^2\theta \qquad \Gamma^r_{tt} = -re^{-2\Lambda+2\Phi}\Phi_{,r} \qquad \Gamma^r_{rr} = \Lambda_{,r} \qquad \Gamma^r_{\theta\theta} = -re^{-2\Lambda} \\
&\Gamma^\theta_{\phi\phi} = \frac{1}{2}\sin 2\theta \qquad \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r} \\
&\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{r} \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta}
\end{aligned}$$

The Riemann tensor is given by

$$\begin{aligned}
R^\alpha_{\beta\mu\nu} &= \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu} \\
R_{\alpha\beta\mu\nu} &= g_{\lambda\alpha}(\Gamma^\lambda_{\beta\mu,\nu} + \Gamma^\lambda_{\beta\nu,\mu} - \Gamma^\lambda_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\lambda_{\sigma\nu}\Gamma^\sigma_{\beta\mu})
\end{aligned}$$

Explicitly for R_{trtr} we get

$$\begin{aligned}
R_{trtr} &= g_{tt}R^t_{rtr} \\
&= -e^{2\Phi}\left[\Gamma^r_{rt,r} + \cancelto{0}{\Gamma^r_{rr,t}} + \Gamma^r_{\sigma r}\Gamma^\sigma_{rr} - \Gamma^r_{\sigma r}\Gamma^\sigma_{rt}\right] \\
&= -e^{2\Phi}\left[\Phi_{,rr} + \Gamma^r_{rr}\Gamma^r_{rr} - \Gamma^r_{rr}\Gamma^r_{rt}\right] \\
&= -e^{2\Phi}\left[\Phi_{,rr} + (\Phi_{,r})^2 - \Phi_{,r}\Lambda_{,r}\right]
\end{aligned}$$

The rest of the components can be similarly calculated [2]

$$\begin{aligned}
R_{trrt} &= \left(\left(\Lambda_{,r} - \Phi_{,r}\right)\Phi_{,r} - \Phi_{,rr}\right)e^{2\Phi} \\
R_{trtr} &= \left(-\left(\Lambda_{,r} - \Phi_{,r}\right)\Phi_{,r} + \Phi_{,rr}\right)e^{2\Phi}
\end{aligned}$$

[1] This was solved mostly using Cadabra. `https://www.physics.drexel.edu/~pgautam/courses/PHYS631/einstein-tensor-expanding-universe.html`

[2] I did this using Cadabra. The detail of this exercise is at `https://www.physics.drexel.edu/~pgautam/courses/PHYS631/HW4Schutz6.35.html`

$$
\begin{aligned}
R_{t\phi t\phi} =& re^{-2\Lambda+2\Phi}\sin^2\theta\Phi_{,r}\\
R_{t\theta t\theta} =& re^{-2\Lambda+2\Phi}\Phi_{,r}\\
R_{t\phi\phi t} =& -re^{-2\Lambda+2\Phi}\sin^2\theta\Phi_{,r}\\
R_{t\theta\theta t} =& -re^{-2\Lambda+2\Phi}\Phi_{,r}\\
R_{r\phi r\phi} =& r\sin^2\theta\Lambda_{,r}\\
R_{r\theta r\theta} =& r\Lambda_{,r}\\
R_{rtrt} =& \left(-\left(\Lambda_{,r}-\Phi_{,r}\right)\Phi_{,r}+\Phi_{,rr}\right)e^{2\Phi}\\
R_{r\phi\phi r} =& -r\sin^2\theta\Lambda_{,r}\\
R_{r\theta\theta r} =& -r\Lambda_{,r}\\
R_{rttr} =& \left(\left(\Lambda_{,r}-\Phi_{,r}\right)\Phi_{,r}-\Phi_{,rr}\right)e^{2\Phi}\\
R_{\theta rr\theta} =& -r\Lambda_{,r}\\
R_{\theta r\theta r} =& r\Lambda_{,r}\\
R_{\theta\phi\theta\phi} =& \frac{1}{2}r^2\left(e^{2\Lambda}\sin(2\theta)(\tan\theta)^{-1}-2e^{2\Lambda}\cos(2\theta)+\cos(2\theta)-1\right)e^{-2\Lambda}\\
R_{\theta t\theta t} =& re^{-2\Lambda+2\Phi}\Phi_{,r}\\
R_{\theta\phi\phi\theta} =& r^2\left(1-e^{2\Lambda}\right)e^{-2\Lambda}\sin^2\theta\\
R_{\theta tt\theta} =& -re^{-2\Lambda+2\Phi}\Phi_{,r}\\
R_{\phi rr\phi} =& -r\sin^2\theta\Lambda_{,r}\\
R_{\phi\theta\theta\phi} =& r^2\left(1-e^{2\Lambda}\right)e^{-2\Lambda}\sin^2\theta\\
R_{\phi r\phi r} =& r\sin^2\theta\Lambda_{,r}\\
R_{\phi\theta\phi\theta} =& r^2\left(e^{2\Lambda}-1\right)e^{-2\Lambda}\sin^2\theta\\
R_{\phi t\phi t} =& re^{-2\Lambda+2\Phi}\sin^2\theta\Phi_{,r}\\
R_{\phi tt\phi} =& -re^{-2\Lambda+2\Phi}\sin^2\theta\Phi_{,r}
\end{aligned}
$$

These are the non zero components of Riemann tensor. □

5. **(Schutz 7.7)** Consider the following four different metrics, as given by their line elements:

   i. $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$;

   ii. $ds^2 = -(1-2M/r)\,dt^2 + (1-2M/r)^-1dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ where M is a constant.

   iii. $ds^2 = -dt^2 + R^2(t)\left[(1-kr^2)^-1dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$, where k is a constant and $R(t)$ is an arbitrary function of t alone.

   (a) For each metric find as many conserved components p_α of a freely falling particle's four momentum as possible.
   Solution:
   The rate of change of momentum is given by

$$
m\frac{\mathrm{d}p_\beta}{\mathrm{d}\tau} = \frac{1}{2}g_{\mu\alpha,\beta}p^\nu p^\alpha
$$

   The momentum p_β is conserved when $g_{\mu\alpha,\beta} = 0$. From the given metric the conserved quantities are

$$
\begin{aligned}
\text{for } i. : &\quad p_t, p_x, p_y, p_z\\
\text{for } ii. : &\quad p_t, p_\phi\\
\text{for } iii. : &\quad p_\phi
\end{aligned}
$$

□

(b) Write i. in the form

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

From this argue that ii. iii. are spherically symmetric. Does this increase the number of conserved components of p_α?

Solution:

The coordinate transformation from Cartesian to polar is

$$\begin{aligned}
x &= r\sin\theta\cos\phi &&\implies dx = \sin\theta + \cos\phi dr + +r\cos\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi\\
y &= r\sin\theta\sin\phi &&\implies dy = \sin\theta + \sin\phi dr + +r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi\\
z &= r\cos\theta &&\implies dz\cos\theta dr = -\sin\theta d\theta
\end{aligned}$$

Substituting these in the line element we get

$$\begin{aligned}
dl^2 =& -dt^2 + dr^2(\sin^2\theta\sin^2\phi\sin^2\theta\cos^2\phi + \cos^2)+\\
&+ d\theta^2\left(r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\cos^2\phi + r^2\sin^2\theta\right)\\
&+ d\phi^2\left(r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi\right)\\
&= -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)
\end{aligned}$$

This is the required transformation in spherical form. □

(c) It can be shown that for ii. and iii. a geodesic that begins with $\theta = \frac{\pi}{2}$ and $p^\theta = 0$- i.e., one which begins tangent to the equatorial plane- always has $\theta = \frac{\pi}{2}$ and $p^\theta = 0$. For these cases use the equation $\vec{p}\cdot\vec{p} = -m^2$ to solve for p^r in terms of m, other conserved quantities, and known functions of position.

Solution:

Expanding the relation $\vec{p}\cdot\vec{p} = -m^2$ we get

$$-m^2 = g_{tt}\left(p^t\right)^2 + g_{rr}\left(p^r\right)^2 + g_{\theta\theta}\left(p^\theta\right)^2 + g_{\phi\phi}\left(p^\phi\right)^2$$

Given $\theta = \pi/2$ and $p^\theta = 0$ we get

$$p^r = \sqrt{\frac{-m^2 - g_{tt}\left(p^t\right)^2 + g_{\phi\phi}\left(p^\phi\right)^2}{g_{rr}}}$$

Since p^t and p^ϕ are conserved substituting the corresponding metric values $g_{\alpha\beta}$ gives the quantity p^r

$$\text{for } ii.;\qquad p^r = \sqrt{\frac{-m^2 + \frac{1-2M}{r}\left(p^t\right)^2 + r^2\left(p^\phi\right)^2}{1 - 2M/r}}$$

$$\text{for } iii.;\qquad p^r = \sqrt{\frac{1-kr^2}{R^2(t)}\left(-m^2 + \left(p^t\right)^2 + (R(t)r)^2\left(p^\phi\right)^2\right)}$$

These are the required expression for p^r in terms of conserved quantities. □

(d) For iii., spherical symmetry implies that if a geodesic begins with $p^\theta = p^\phi = 0$, these remain zero. Use this to show that when $k = 0$, p_r is a conserved quantity.

Solution:

The rate of change of momentum is given by

$$\begin{aligned}
m\frac{\mathrm{d}p_\beta}{\mathrm{d}\tau} &= \frac{1}{2}g_{\mu\alpha,\beta}p^\nu p^\alpha\\
m\frac{\mathrm{d}p_r}{\mathrm{d}\tau} &= \frac{1}{2}\left(g_{tt,r}(p^t)^2 + g_{rr,r}(p^r)^2 + g_{\theta\theta,r}(p^\theta)^2 + g_{\phi\phi}(p^\phi)^2\right)
\end{aligned}$$

But for $k = 0$, $g_{rr,r} = 0$ and $g_{tt,r} = 0$ and given $p^\theta = p^\phi = 0$ we get

$$m\frac{\mathrm{d}p_r}{\mathrm{d}\tau} = 0$$

This proves that p_r is a conserved quantity. □

6. What fractional energy does a photon lose if it goes from the surface of the earth to deep space?
Solution:
When the photon goes from the surface of earth to outer space, it must lose the gravitational potential energy that is has near the surface of earth. So the photon must lose this energy. For photon

$$(U^0)^2 g_{00} = -1$$

On surface of earth with weak field limit

$$g_{00} = -(1 - 2\phi)$$

So near the surface of earth

$$U^0 \simeq 1 + \phi$$

In far space metric Minkowski $g_{00} = -1$ so in far space

$$U^0 = 1$$

So ratio of energy

$$\frac{1}{1+\phi} = 1 - \phi$$

So change in energy is $\sim \phi$ On the surface of earth the gravitational potential is

$$\phi = -\frac{GM}{c^2 r} = -\frac{6.672 \times 10^{-11} \times 6.0 \times 10^{24}}{6.4 \times 10^6 \times 9 \times 10^{16}} \approx 7 \times 10^{-10}$$

So the photon must lose this energy fractionally. $\square$