PHYS 517: Quantum Mechanics II

Homework #1

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1. **(Sakurai 2.33)** The propagator in momentum space is given by $\langle \mathbf{p}'', t|\mathbf{p}', t_0\rangle$. Derive an explicit expression for $\langle \mathbf{p}'', t|\mathbf{p}', t_0\rangle$ for the free particle case.
Solution:
For a free particle the Hamiltonian is

$$H = \frac{\mathbf{p}^2}{2m}$$

So the time evolution operator for any state in momentum space is given by

$$\mathcal{U}(t) = e^{\frac{iHt}{\hbar}} \qquad \Rightarrow \qquad \exp\left[\frac{i\mathbf{p}^2 t}{2m\hbar}\right]$$

The base kets evolve over time as

$$|\mathbf{p}', t\rangle = \mathcal{U}(t)^\dagger\, |\mathbf{p}', 0\rangle \qquad \to \langle \mathbf{p}', t| = \langle \mathbf{p}', 0|\,\mathcal{U}(t) = \langle \mathbf{p}', t_0|\exp\left[\frac{i\mathbf{p}^2 t}{2m\hbar}\right]$$

So the propagator becomes

$$\begin{aligned}
\langle \mathbf{p}'', t|\mathbf{p}', t_0\rangle &= \langle \mathbf{p}'', 0|\exp\left[\frac{i{p''}^2 t}{2m\hbar}\right]\exp\left[-\frac{i{p'}^2 t_0}{2m\hbar}\right]|\mathbf{p}', 0\rangle \\
&= \exp\left[\frac{i}{2m\hbar}\left(p''^2 t - p'^2 t_0\right)\right]\langle \mathbf{p}'', 0|\mathbf{p}', 0\rangle \\
&= \exp\left[\frac{i}{2m\hbar}\left(p''^2 t - p'^2 t_0\right)\right]\delta(\mathbf{p}'' - \mathbf{p}')
\end{aligned}$$

This gives explicit expression for the propagator of the free particle. □

2. **(Skurai 2.37)**

   (a) Verify $[\Pi_i, \Pi_j] = \left(\frac{i\hbar e}{c}\right)\varepsilon_{ijk}B_k$. and $m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \frac{\mathrm{d}\Pi}{\mathrm{d}t} = e\left[\mathbf{E} + \frac{1}{2c}\left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\times\mathbf{B} - \mathbf{B}\times\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)\right]$
   Solution:
   The kinematical momentum for electromagnetic field is defined as $\mathbf{\Pi} \equiv m\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{p} - \frac{e\mathbf{A}}{c}$ where $\mathbf{A}$ is the vector magnetic potential is a function of operator $\mathbf{x}$. The commutator then is

$$\begin{aligned}
[\Pi_i, \Pi_j] &= \left[p_i - \frac{e}{c}A_i, p_j - \frac{e}{c}A_j\right] \\
&= [p_i, p_j] - \left[p_i, \frac{e}{c}A_j\right] - \left[\frac{e}{c}A_i, p_j\right] + \left[\frac{e}{c}A_i, \frac{e}{c}A_j\right] \\
&= 0 - \frac{e}{c}\left(-i\hbar\frac{\partial A_j}{\partial x_i}\right) - \frac{e}{c}\left(i\hbar\frac{\partial A_i}{\partial x_j}\right) + 0 \\
&= \frac{i\hbar e}{c}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right) \\
&= \frac{i\hbar e}{c}B_k
\end{aligned}$$

repeating this same process for all the components of this kinematical momentum operator we get

$$[\Pi_i, \Pi_j] = \frac{i\hbar e}{c}\varepsilon_{ijk}B_k \tag{1}$$

The Hamiltonian for electromagnetic field id $H = \frac{\Pi^2}{2m} + e\phi$. For the Lorentz force formula we have $m\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \equiv \mathbf{\Pi}$ differentiating this with time gives $m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \frac{\mathrm{d}\mathbf{\Pi}}{\mathrm{d}t}$ by using Heisenberg equation of motion we can write

$$\begin{aligned}
m\frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} = \frac{\mathrm{d}\Pi_i}{\mathrm{d}t} &= \frac{1}{i\hbar}[\Pi_i, H] \\
&= \frac{1}{i\hbar}\left[\Pi_i, \frac{\Pi^2}{2m} + e\phi\right] \\
&= \frac{1}{i\hbar}\left[\Pi_i, \frac{\mathbf{\Pi}^2}{2m}\right] + \frac{1}{i\hbar}\left[p_i + \frac{e}{c}A_x, e\phi\right] \\
&= \frac{1}{2mi\hbar}\sum_j \left[\Pi_i, \Pi_j^2\right] + \frac{1}{i\hbar}[p_i, e\phi]
\end{aligned}$$

But the commutator of $\left[\Pi_i, \Pi_j^2\right] = \Pi_r[\Pi_i, \Pi_j] + [\Pi_i, \Pi_j]\Pi_r$ which by use of (1) reduces to

$$\left[\Pi_i, \Pi_j^2\right] = \Pi_j\frac{i\hbar e}{c}\varepsilon_{ijk}B_k + \frac{i\hbar e}{c}\varepsilon_{ijk}B_k\Pi_j$$

And also $\frac{1}{i\hbar}[p_i, e\phi] = \frac{1}{i\hbar}(-i\hbar)\frac{\partial e\phi}{\partial x} = -eE_i$

Using these two facts back in in the original commutator leads to

$$\begin{aligned}
m\frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} &= \frac{1}{2mi\hbar}\sum_j \varepsilon_{ijk}p_jB_k\frac{i\hbar e}{c} + \varepsilon_{ijk}B_k\frac{i\hbar e}{c}p_j - eE_i \\
&= e\left[E + \frac{1}{2c}\sum_j\left(\frac{\mathrm{d}x_j}{\mathrm{d}t}B_k - B_j\frac{\mathrm{d}x_k}{\mathrm{d}t}\right)\right]
\end{aligned}$$

The above expression can be obtained for each components ij and k to obtain the required relation in 3D

$$m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \frac{\mathrm{d}\Pi}{\mathrm{d}t} = e\left[\mathbf{E} + \frac{1}{2c}\left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\times\mathbf{B} - \mathbf{B}\times\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)\right]$$

This is the required lorentz force relation. □

(b) Verify $\frac{\partial\rho}{\partial t} + \nabla'\cdot\mathbf{j} = 0$ with $\mathbf{j}$ given by $\mathbf{j} = \left(\frac{\hbar}{m}\,\mathrm{Im}(\psi^*\nabla'\psi) - \left(\frac{e}{mc}\right)\mathbf{A}|\psi|^2,\right)$

Solution:

By definition the probability density function is the absolute value square of waefunction. The Hamiltonian for electromagnetic field for arbitrary wavefunction ψ is given by

$$H = \frac{\Pi^2}{2m} + e\phi = \frac{2}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + e\phi$$

The momentum operator in position space wavefunction can be written as $-i\hbar\nabla$. Using the schrodinger equation $H\psi = E\psi$ where operator E is given by $E = i\hbar\frac{\partial}{\partial t}$ we get

$$\begin{aligned}
H\psi &= i\hbar\frac{\partial}{\partial t}\psi \\
\frac{\partial\psi}{\partial t} &= \frac{1}{i\hbar}\left[\frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)^2 + e\phi\right] \\
&= \frac{1}{i\hbar}\left[\frac{-\hbar^2}{2m}\nabla^2 + i\hbar\frac{e}{2mc}\left(\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\right) + \frac{e^2}{2mc^2}A^2 + e\phi\right]\psi \\
&= \frac{1}{i\hbar}\left[\frac{-\hbar^2}{2m}\nabla^2\psi + i\hbar\frac{e}{2mc}\left(\nabla\cdot(\mathbf{A}\psi) + \mathbf{A}\cdot\nabla\psi\right) + \frac{e^2}{2mc^2}A^2\psi + (e\phi)\psi\right] \\
&= \frac{i\hbar}{2m}\nabla^2\psi + \frac{e}{2mc}\left(\nabla\cdot\mathbf{A}\right)\psi + \frac{e}{2mc}\mathbf{A}\cdot\nabla\psi + \frac{e}{2mc}\mathbf{A}\cdot\nabla\psi + \frac{-i}{\hbar}\left(\frac{e^2}{2mc^2}A^2 + e\phi\right)\psi \\
&= \frac{i\hbar}{2m}\nabla^2\psi + \frac{e}{2mc}\left(\nabla\cdot\mathbf{A}\right)\psi + \frac{e}{mc}\mathbf{A}\cdot\nabla\psi + \frac{-i}{\hbar}\left(\frac{e^2}{2mc^2}A^2 + e\phi\right)\psi
\end{aligned}$$

Taking the conjugate of this expression leads to

$$\frac{\partial\psi^*}{\partial t} = \frac{-i\hbar}{2m}\nabla^2\psi^* + \frac{e}{2mc}\left(\nabla\cdot A\right)\psi^* + i\hbar\frac{e}{mc}\mathbf{A}\cdot\nabla\psi + \frac{i}{\hbar}\left(\frac{e^2}{2mc^2}A^2 + e\phi\right)\psi^* \tag{2}$$

Taking the time derivative of the probability density function we get

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial t} = \frac{\partial}{\partial t}(\psi^*\psi) = \psi^*\frac{\partial\psi}{\partial t} + \frac{\partial\psi^*}{\partial t}\psi$$

For a divergence free magnetic vector potential (which we can always choose), Multiplying (2) by ψ and its conjugate by ψ^* and adding we get

$$\begin{aligned}
\frac{\partial\rho}{\partial t} &= \psi^*\frac{i\hbar}{2m}\nabla^2\psi + \psi^*\frac{e}{mc}\mathbf{A}\cdot(\nabla\psi) + \psi\frac{-i\hbar}{2m}\nabla^2\psi^* + \psi\frac{e}{mc}\mathbf{A}\cdot(\nabla\psi^*) \\
&= \frac{i\hbar}{2m}\left[\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right] + \frac{e}{mc}\left(\psi\mathbf{A}\cdot(\nabla\psi^*) + \psi^*\mathbf{A}\cdot(\nabla\psi)\right) \\
&= \frac{i\hbar}{2m}(2i\nabla\cdot\mathrm{Im}(\psi^*\nabla\psi) + \frac{e}{mc}(\nabla\cdot(\mathbf{A}\psi^*\psi)) \\
&= -\frac{\hbar}{m}\nabla\cdot(\mathrm{Im}(\psi^*\nabla\psi)) + \frac{e}{mc}\nabla\cdot(\mathbf{A}|\psi|^2) \\
&= -\nabla\cdot\left(\frac{\hbar}{m}\mathrm{Im}(\psi^*\nabla\psi) - \frac{e}{mc}\mathbf{A}|\psi|^2\right) \\
&= -\nabla\cdot\mathbf{j}
\end{aligned}$$

This completes the proof. □

3. **(Sakurai 2.38)** Consider a Hamiltonian of the spinless particle of charge e. In presence of a static magnetic field, the interaction terms can be generated by

$$\mathbf{P}_{\text{operator}} \to \mathbf{P}_{\text{operator}} - \frac{e\mathbf{A}}{c},$$

where $\mathbf{A}$ is the appropirate vector potential. Suppose, for simplicity, the magnetic field $\mathbf{B}$ is uniform in the positive $z-$ direction. Prove that the above presciprition indeed leads to the correct expression for the interaction of the orbital magnetic moment $(e/2mc)\,\mathbf{L}$ with the magnetic field $\mathbf{B}$. Show that there is also an extra term proportional to $B^2(x^2+y^2)$, and comment briefly on its physical significance.

Solution:
Since the electric field is zero we can assign a scalar potential as constant and the constant can always be chosen 0 thus $\phi = 0$. The vector magnetic potential for uniform magnetic field is $\mathbf{A} = \frac{1}{2}\mathbf{x}\times B\hat{\mathbf{z}}$. Since there

is a free choice of vector magnetic potential as long as its curl is divergence free, we chose this potential which is also divergence free. Thus for this case $\nabla \cdot A = 0$.

From (2) we have the hamiltonian of the system is

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{i\hbar e}{mc}\mathbf{A}\cdot\nabla + \frac{i\hbar e}{2mc}\nabla\cdot\mathbf{A} + \frac{e^2}{2mc^2}\mathbf{A}^2$$

Since $\nabla \cdot \mathbf{A} = 0$ by our choice the interaction operator terms introduced due to the presence of magnetic potential is

$$\frac{i\hbar e}{mc}\mathbf{A}\cdot\nabla + \frac{e^2}{2mc^2}\mathbf{A}^2 = -\frac{e}{mc}\mathbf{A}\cdot(-i\hbar\nabla) + \frac{e^2}{2mc^2}\mathbf{A}^2$$

But the operator $-i\hbar\nabla$ is the momentum operator $\mathbf{p}$ and $\mathbf{A}^2 = \frac{1}{4}B^2\left(x^2+y^2\right)$ This enables us to write the interaction terms as

$$-\frac{e}{mc}B\frac{1}{2}\left(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}\right)\cdot\mathbf{p} + \frac{e^2B^2}{8mc^2}\left(x^2+y^2\right)$$

We can recocnize the term $(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}})\cdot\mathbf{p} = -yP_x + xP_y = L_z$ Substuting this in the above expression we get

$$\frac{e}{2mc}BL_z + \frac{e^2B^2}{8mc^2}\left(x^2+y^2\right)$$

So the final hamiltonian becomes

$$H = \left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{e}{2mc}BL_z + \frac{e^2B^2}{8mc^2}\left(x^2+y^2\right)\right]$$

So the interaction terms introduced in the absence of scalar potential but the presence of magnetic potential has operator for orbital angular momentum $\frac{e}{2mc}BL_z$ and a term proportional to $B^2\left(x^2+y^2\right)$ □

4. **(Sakurai 2.39)** An electron moves in he presence of a uniform magnetic field in the z-direction ($\mathbf{B} = B\hat{\mathbf{z}}$)

   (a) Evaluate $[\Pi_x, \Pi_y]$ where $\Pi_x \equiv p_x - \dfrac{eA_x}{c}, \qquad \Pi_y \equiv p_y - \dfrac{eA_y}{c}$.

   Solution:

$$\begin{aligned}
[\Pi_x, \Pi_y] &= \left[p_x - \frac{e}{c}A_x, p_y - \frac{e}{c}A_y\right] \\
&= [p_x, p_y] - \left[p_x, \frac{e}{c}A_y\right] - \left[\frac{e}{c}A_i, p_j\right] + \left[\frac{e}{c}A_i, \frac{e}{c}A_j\right] \\
&= 0 - \frac{e}{c}\left(-i\hbar\frac{\partial A_j}{\partial x}\right) - \frac{e}{c}\left(i\hbar\frac{\partial A_x}{\partial x_y}\right) + 0 \\
&= \frac{i\hbar e}{c}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&= \frac{i\hbar e}{c}B_z
\end{aligned}$$

   Which is the required expression for the comutator. □

   (b) By comparing the Hamiltonian and the commutation relation obtained in 4 with those of the one-dimensional oscillator problem, show how we can immediately write the energy eigenvalues as

$$E_{k,n} = \frac{\hbar^2k^3}{2m} + \left(\frac{|eB|\hbar}{mc}\right)\left(n + \frac{1}{2}\right)$$

   Solution:

Since the charged particle is only in the magnetic field, the electric field is absent, which means the electric potential is a constant whiich we may assume to be 0. So the hamiltonian of the system is

$$H = \frac{\mathbf{\Pi}^2}{2m} = \frac{\Pi_z^2}{2m} + \frac{\Pi_y^2}{2m} + \frac{\Pi_x^2}{2m}$$

The energy eigenvalue equation for the a general wavefunction $\psi_\alpha(x')$ we have

$$H\psi_\alpha(x') = \left[\frac{\Pi_z^2}{2m} + \frac{\Pi_y^2}{2m} + \frac{\Pi_x^2}{2m}\right]\psi_\alpha(x')$$

Since the magnetic field is completely in $\hat{\mathbf{z}}$ the vector magnetic potential can be written as $A(\mathbf{x}) = \frac{1}{2}\mathbf{x} \times B\hat{\mathbf{z}}$ so that $A_z = 0$. This simplifies the eigenvalue equation to

$$H\psi_\alpha(x') = \left[\frac{p_z^2}{2m} + \frac{\Pi_y^2}{2m} + \frac{\Pi_x^2}{2m}\right]\psi_\alpha(x')$$

The first of these three expression p_z has known eigenvalue $\hbar k$ given in the problem. The second two terms can be evaluated using the One dimensional simple harmonic oscillator. Since the comutator $[\Pi_x, \Pi_y] = i\hbar\frac{e}{c}B$ we can scale Π_y by $\frac{c}{eB}$ to make $\left[\Pi_x, \frac{c}{eB}\Pi_y\right] = i\hbar$. Let $Y = \frac{c}{eB}\Pi_y$ Using this the expression becomes

$$H\psi_\alpha(x') = \left[\frac{p_z^2}{2m} + \frac{\Pi_x^2}{2m} + \frac{1}{2}m\frac{e^2B^2}{m^2c^2}Y^2\right]\psi_\alpha(x')$$

We can again try the raising a operator and lowering operators $a^\dagger$ out of the last two expression.

$$a = \sqrt{\frac{eB}{2\hbar c}}\left(Y + \frac{ic}{eB}\Pi_x\right) \qquad a^\dagger = \sqrt{\frac{eB}{2\hbar c}}\left(Y - \frac{ic}{eB}\Pi_x\right)$$

And since $a^\dagger a = \frac{mc}{\hbar eB}H + \frac{i}{2\hbar}[Y, \Pi_x] = \frac{Hmc}{\hbar eB} - \frac{1}{2}$. In complete analogy to SHO we find $a^\dagger a$ works as simultaneous operator with Hamiltonian H, i.e. $a^\dagger a$ commutes with H, and so acts on energy eigenstates to give integer n as its eigenvalue. So the eigenvalue become

$$H\psi_\alpha(x') = \left[\frac{p_z^2}{2m}\psi_\alpha(x')\right] + \left[\frac{\Pi_x^2}{2m} + \frac{1}{2}m\frac{e^2B^2}{m^2c^2}Y^2\right]\psi_\alpha(x')$$
$$H\psi_\alpha(x') = \frac{\hbar^2k^2}{2m}\psi_\alpha(x') + \left[\left(n + \frac{1}{2}\right)\hbar\frac{|eB|}{mc}\right]\psi_\alpha(x')$$

So the eigenvalue of the operator H which are the energy values are

$$E_n = \frac{\hbar^2k^2}{2m} + \left(n + \frac{1}{2}\right)\hbar\frac{|eB|}{mc}$$

This gives the allowed energy of the charged particle. □