Renormalization Group Theory and the 2 Dimensional Ising Problem

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1 Introduction

When looking for a solution to a given physical problem, the local view generally prevails. An integral or sum is performed over a set length scale in the system, and the result is used to describe the system's behavior. For statistical and quantum mechanical problems, such a view is generally quite successful: the system is described locally, and longer range interactions are unimportant. However, near phase transitions, all length scales become important to the behavior of the system, and the strict local view breaks down. The goal of the renormalization group is to connect these small and large scale fluctuations.

2 The renormalization group

The first description of the "renormalization group" was given in 1953 by Stueckelberg and Petermann [2], with elaboration in 1954 by Gell-Mann and Low [3]. This methodology was used to make otherwise divergent calculations in QED converge. Wilson followed the lead of Kadanoff in developing a nearest-neighbor description which used the power of the renormalization group to solve the Ising problem [5]. It was for this work that Wilson received the Nobel Prize in 1983. A more detailed description of the development of the theory can be found in his Nobel paper [6].

The renormalization group itself allows one to solve systems where fluctuations on a wide range of length scales are important, as is the case in critical phenomena. It is a method for accurately connecting the small scale variations with larger and larger scales. The name follows from the renormalization procedure for Feynman diagrams. However, the group representation of the transformation is unused in most applications of the theory. "The fact that the transformation can be described as a group (by adding inverses and an identity element) plays no role." [7]

2.1 Definition

Let $H = H_{\{\sigma_l\}}(\{K_\alpha\})$ be the Hamiltonian for a system with parameters $\{K_\alpha\}$ (most of which are 0), spins $\{\sigma_l\}$, lattice spacing $l > 1$ and dimension d. The set of parameters $\{K_\alpha\}$ form a vector $\mathbf{K}$ in some vector space $\mathcal{V}$. Consider a transformation $\mathcal{R}_l$ which reduces the number of degrees of freedom from N to N' and the correlation length from ξ to ξ' as follows:

$$N' = l^{-d}N, \quad \xi' = l^{-1}\xi. \tag{2.1}$$

This transformation takes $\{K_\alpha\} \to \{K'_\alpha\} \in \mathcal{V}$. The new parameter set $\{K'_\alpha\}$ will have more non-zero elements than the initial set (this will be shown in detail later). Thus,

$$\mathbf{K}' = \mathcal{R}_l(\mathbf{K}). \tag{2.2}$$

By repeatedly applying $\mathcal{R}_l$ to an element of $\mathcal{V}$ we create a sequence of $n + 1$ vectors

$$\mathbf{K}^{(n)} = \mathcal{R}_l(\mathbf{K}^{(n-1)}) = \cdots = \mathcal{R}_l(\mathbf{K}^{(0)} = \mathbf{K}),$$

where $\mathbf{K}^{(n)}$ has a new correlation length after repeated application of 2.1,

$$\xi^{(n)} = l^{-n}\xi. \tag{2.3}$$

The transformation 2.2, is called the renormalization group operator. Its exact form depends on the system under consideration. The power of the renormalization group comes in its creation of fixed points, where

$$\mathbf{K}^* = \mathcal{R}_l(\mathbf{K}^*). \tag{2.4}$$

At such a fixed point, $\xi(\mathbf{K}^*) = l^{-1}\xi(\mathbf{K}^*)$, either $l = 0$, $\xi = 0$, or $\xi = \infty$. Since $l > 0$ as defined above, and the case $\xi = 0$ is uninteresting, we are left with one possibility: ξ must be infinite. An infinite correlation length is known to signify a critical point, or phase transition, as discussed in the introduction.

By use of this transformation, we can determine the value of such a fixed point by iterating through a series of such transformations and looking for fixed points in the resulting sum. For example, in a spin lattice, with free-energy

$$e^{-\beta A} = \sum_{\{\sigma_i\}} e^{-\beta H_{\sigma_i}(\mathbf{K})} \quad \alpha = 1, 2, \ldots,$$

we can apply the transformation $\mathcal{R}_l$ to get a new state, as above. Writing the new state in a similar form to the old state,

$$e^{-\beta A} = e^{N'K'_0} \sum_{\{\sigma'_j\}} e^{-\beta H_{\{\sigma'_j\}}(\mathbf{K}')},$$

where the elements of $\mathbf{K}'$ come from 2.2, and the new spin state of the system is represented by $\{\sigma'_j\}$. This new configuration will have a free-energy per spin of

$$f(\mathbf{K}) = l^{-d}(-K'_0 + f(\mathbf{K}')),$$

so we have introduced a new parameter K'_0 and transformed to a new vector $\mathbf{K}'$. As an example of this transformation, we will consider the solution to the 2-dimensional Ising model.

3 The Ising model

Consider a system of spins, all aligned in the z-direction. Each element of this system has spin σ_i= +1 for "up" spin or -1 for "down" spin and all neighboring elements have an interaction energy J. The internal energy of this system, is thus

$$E = const. - J \sum_{i,j} \sigma_i \sigma_j, \tag{3.1}$$

where the sum is over all nearest neighbors i and j. This is the *Ising Model*, a simplification of the general case where the sum is over vector spins $\mathbf{s}_i \cdot \mathbf{s}_j$. This model is used, first because it is solvable, and second because it has many analogs in other systems involving phase transitions.

In the 1-dimensional Ising model, the spin system is a loop, where we consider the interactions between neighbors as one moves around the loop. The neighbors of the Nth element are the N-1 and 1st elements. The loop eliminates the end effects which would otherwise mar the calculation, but it does not affect the properties of an infinite length chain of elements. There are several methods for solving this system which are explored in chapter 12 of [1], and they will not be described here. However, the 2 dimensional lattice version of this problem is much more difficult, for which we shall need the power of the renormalization group.

3.1 Solving the 2D Ising model

Consider the extension of the previous model to a two dimensional lattice of size $N \times N$. We again only consider the interaction between nearest neighbors, which produces a partition function as one would expect,

$$Q_N(T) = \sum_{\{\sigma_n\}} exp(\sum_{\mathbf{n}} \sum_{\mathbf{i}} K \sigma_{\mathbf{n}} \sigma_{\mathbf{n+i}}) \quad (K = \beta J), \tag{3.2}$$

where the sum in the exponent is over all nearest neighbors, as in 3.1. $\mathbf{n}$ is a vector with integral components that represents a site in the lattice, and $\mathbf{i}$ is a unit vector in the direction ($\hat{i} = 1$ or 2). Note that this ignores the diagonal interactions. These interactions will be accounted for during in successive applications of the renormalization group.

The usual method for solving such a system is to determine the value of the exponent numerically and add up all such configurations. However, near the critical temperature, the correlation length becomes of similar scale to the size of the whole system. Thus, in order to achieve the correct result, we must let N be of similar scale to the size of the entire system. This is not a feasible
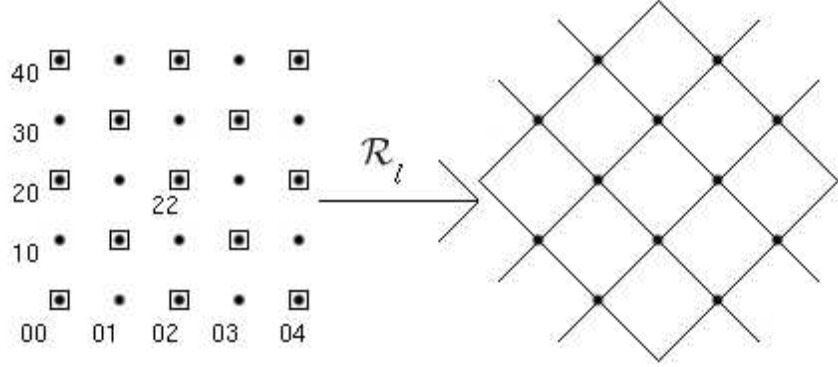
Figure 1: An example 2d Ising spin lattice. The boxed points are summed over and the set has only the unboxed points remaining, creating a new lattice. The initial lattice is labeled by $\sigma_{\mathbf{n}}$ while the new lattice is labeled by $\tau_{\mathbf{m}}$.

calculation, since the number of operations in the sum is $O(2^{N^2})$. Thus, we will leverage the power of the renormalization group to reduce the calculation to a series of workable steps at each length scale, and combine the results from each step to get the final answer.

To express the new state of of the system, $\mathcal{H}_1(\tau)$, with the renormalized spin lattice τ, we write the Kadanoff transformation

$$e^{\mathcal{H}_1(\tau)} \quad = \quad \sum_{\{\sigma_{\mathbf{n}}\}} (\prod_{\mathbf{m}} \delta_{\tau_{\mathbf{m}} \sigma_{\mathbf{n}(\mathbf{m})}}) e^{\mathcal{H}_0(\sigma)}. \tag{3.3}$$

The notation, taken from [4] is somewhat confusing, so a short explanation is in order. The delta-function $\delta_{\tau_{\mathbf{m}} \sigma_{\mathbf{n}(\mathbf{m})}}$ ensures that half the elements remain fixed: $\mathbf{n}(\mathbf{m})$ gives the index of $\sigma_{\mathbf{n}}$ in the new lattice ($\sigma_{\mathbf{n}} \to \tau_{\mathbf{m}}$). $\mathcal{H}_0(\sigma)$ is the element in the exponent of the partition function 3.2, and $\mathcal{H}_1(\tau)$ is the equivalent element for the new lattice.

This transformation naturally provides a recursion formula from step $i - 1$ to step i,

$$e^{\mathcal{H}_i(\tau)} = \sum_{\{\sigma_{\mathbf{n}}\}} (\prod_{\mathbf{m}} \delta_{\tau_{\mathbf{m}} \sigma_{\mathbf{n}(\mathbf{m})}}) e^{\mathcal{H}_{i-1}(\sigma)}. \tag{3.4}$$

The spins $\sigma_{\mathbf{n}}$ in the $(i - 1)$st step, become the spins $\tau_{\mathbf{m}}$ in the ith step. At each step in this process, we reduce the number of lattice points by $N^2/2$. If N is finite, we will have summed over all the points after only $O(\log_2 N)$ steps, a dramatic improvement from the standard calculation. If N is infinite, we need only iterate until we reach the desired precision for the calculation. The $N \to \infty$ case will not be considered in this discussion.

4

Let us look at one element of the lattice, σ_{22}, as shown in (fig: 1). This element will be coupled with σ_{12}, σ_{21}, σ_{23} and σ_{32}. Note that these coupled elements will be fixed in the next step, and so we can take this element of the sum, 3.3, independently of all other elements.

$$
\begin{aligned}
S &= \sum_{\sigma_{22}=\pm 1} e^{K\sigma_{12}\sigma_{22}} e^{K\sigma_{21}\sigma_{22}} e^{K\sigma_{23}\sigma_{22}} e^{K\sigma_{32}\sigma_{22}} \\
&= \sum_{\sigma_{22}=\pm 1} e^{K\sigma_{22}(\sigma_{12}+\sigma_{21}+\sigma_{23}+\sigma_{32})} \\
&= 2\cosh K(\sigma_{12}+\sigma_{21}+\sigma_{23}+\sigma_{32}).
\end{aligned}
\tag{3.5}
$$

The 2 comes from the two possible spin states of σ_{22}. All other points give a similar result, so the result of 3.3 will be a product of cosh functions, one for each summed-over lattice point.

What we want is a function with a similar form to the original $\mathcal{H}_0(\sigma)$ with a perturbation. Thus, we will attempt to write 3.5 as a polynomial in $\sigma_{\mathbf{m}}$. First, we write the sum as an exponential with three terms, each dependent on K:

$$
\begin{aligned}
S &= 2\cosh K(\sigma_{12}+\sigma_{21}+\sigma_{23}+\sigma_{32}) \\
&= \exp(A + B(\sigma_{12}(\sigma_{21}+\sigma_{23}+\sigma_{32})+\sigma_{21}(\sigma_{23}+\sigma_{32})+\sigma_{23}\sigma_{32}) \\
&\quad + C(\sigma_{12}\sigma_{21}\sigma_{23}\sigma_{32})).
\end{aligned}
\tag{3.6}
$$

The non-constant terms involve the nearest-neighbor and quadrupole amongst the fixed lattice points.

To show the validity of this new form, note that the sum is symmetric under permutations of $\sigma_{\mathbf{n}}$ and sign changes ($\sigma_{\mathbf{n}} \rightarrow -\sigma_{\mathbf{n}}$ for all $\mathbf{n}$). This leaves three cases,

1. all $\sigma_{\mathbf{n}} = +1$

2. one $\sigma_{\mathbf{n}} = -1$, the remaining three +1

3. two $\sigma_{\mathbf{n}} = -1$ and two +1

which when substituted into 3.6, produces three relations,

$$
\begin{aligned}
2\cosh(4K) &= e^{A+6B+C}, \\
2\cosh(2K) &= e^{A-C}, \\
2\cosh(0K) &= e^{A-2B+C}.
\end{aligned}
$$

Solving these simultaneous equations gives us A, B and C as a function of K,

$$
\begin{aligned}
A(K) &= \ln 2 + \frac{1}{8}(\ln\cosh 4K + 4\ln\cosh 2K), \\
B(K) &= \frac{1}{8}\ln\cosh 4K, \\
C(K) &= \frac{1}{8}(\ln\cosh 4K - 4\ln\cosh 2K,
\end{aligned}
$$

as desired.

We now have a value for each term in 3.6. Substituting this into the Kadanoff transformation 3.3, and transforming $\sigma_{\mathbf{n}} \to \tau_{\mathbf{m}}$, we find

$$
\begin{aligned}
\mathcal{H}_1(\tau) &= \frac{N^2}{2}A(K) + (\sum_{\mathbf{m}}\sum_{\mathbf{i}} 2\tau_{\mathbf{m}}\tau_{\mathbf{m+i}} + \sum_{\mathbf{m}}\sum_{\pm}\tau_{\mathbf{m}}\tau_{\mathbf{m+\hat{1}\pm\hat{2}}})B(K) \quad (3.7)\\
&\quad + \sum_{\mathbf{m}} C(K)\tau_{\mathbf{m}}\tau_{\mathbf{m+\hat{1}}}\tau_{\mathbf{m+\hat{2}}}\tau_{\mathbf{m+\hat{1}+\hat{2}}}.
\end{aligned}
$$

The terms bear a bit of explanation. The constant term is proportional to the area of the new lattice. The first term two-element term involves the new-lattice nearest-neighbor interactions while the second two-element term involves the diagonal nearest-neighbors. The 2 in the first two-element term must be included, because each nearest-neighbor term appears in two original-lattice sums (σ_{22} and σ_{33}, for example). The four-element term is the quadrupole coupling in the new lattice.

We now have an exact formula for the interaction energy of the new lattice. $A(K)$, $B(K)$, and $C(K)$ can be computed, given the particular interaction energy J, and temperature, T, of the system. We can now use the recursion relation 3.4 to find the successive terms of the renormalization group transformation. Unfortunately, the next terms will be more complicated, due to their dependence on more than just the nearest-neighbor couplings. However, the diagonal and four-spin couplings, which arise from 3.7, are generally much smaller than the nearest-neighbor interactions, so a perturbation expansion will suffice for those terms.

So long as the nearest-neighbor interactions are the most important terms in $\mathcal{H}_{i-1}$, we can carry out the transformation and compute the interaction energy at the $\mathcal{H}_i$th step without worry. There are alternate methods which allow strong higher-order interactions, such as one developed by Niemeyer and Van Leeuwen, in 1973. For a sample calculation on a 15-spin lattice, see [4], section VI. His calculation, which was feasible on a computer of the time) produced critical exponents 2.4, to within 0.2% of the measured value.

References

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