Zero-Point Energy and the Casimir Force

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1 Introduction

The Hamiltonian of system of harmonic oscillators,

$$H = \sum_m \hbar\omega_m (a_m^\dagger a_m + \frac{1}{2}), \tag{1}$$

when applied to the modes of the EM field results in an infinite zero-point energy (zero-point is when $a_m^\dagger a_m = 0$). This effect was first explored by Casimir in 1948 [1], but its strange implications have directly measurable effects.

Unfortunately for astronomers, it does not help with the dark energy problem. The observed acceleration of the expansion of the universe can be accounted for by a vacuum energy density of

$$\rho_c = 9 \times 10^{-22}\, J/m^3. \tag{2}$$

There was once some hope that the zero-point energy could provide this "cosmological constant". As we will show, however, the zero-point energy is much to large to account for the cosmological vacuum energy.

2 Vacuum energy density

Maxwell's equations in free space are as follows

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \\ \nabla \cdot \mathbf{E} &= 0, \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned}$$

Following (Ballentine 19.1) we find normal modes of the $\mathbf{E}$ field from the wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

(an identical analysis can be performed for the $\mathbf{B}$ field). The solutions to this equation can be represented as a sum of mode functions

$$\mathbf{E}(\mathbf{x}, t) = \sum_m f_m(t)\mathbf{u_m}(\mathbf{x}). \tag{3}$$

These mode functions $\mathbf{u}_m(\mathbf{x})$ satisfy

$$\begin{aligned} \nabla^2 \mathbf{u}_m(\mathbf{x}) &= -k_m^2 \mathbf{u}_m(\mathbf{x}), && (4) \\ \nabla \cdot \mathbf{u_m}(\mathbf{x}) &= 0, && (5) \\ \hat{\mathbf{n}} \times \mathbf{u_m}(\mathbf{x}) &= 0 \text{ on a conducting surface,} \end{aligned}$$

where $\hat{\mathbf{n}}$ is the unit normal to the conducting surface.

The solution to (eq: 4) in 3-space has vector components

$$\begin{aligned}u_x &= A_1\cos(k_1x)\sin(k_2y)\sin(k_3z),\\ u_y &= A_2\sin(k_1x)\cos(k_2y)\sin(k_3z),\\ u_z &= A_3\sin(k_1x)\sin(k_2y)\cos(k_3z).\end{aligned}$$

The wave vector $\mathbf{k}$ has three components, $k_i = \frac{n_i\pi}{L_i}$ with frequency $\omega_k = c\sqrt{k_1^2+k_2^2+k_3^2}$. The divergence condition (eq: 5) implies $\mathbf{A}\cdot\mathbf{k}=0$. Thus $\mathbf{A}$ and $\mathbf{k}$ are not linearly independent, which means that there are two independent modes A_i for each wave vector $\mathbf{k}$ when all $n_i \neq 0$.

The number of modes in the cavity volume $\Omega = L_1L_2L_3$ is $2\Omega/\pi^3$. Since $\mathbf{k}$ is strictly positive, we only are looking at one octant of phase space. For large $\mathbf{k}$, the zero-point energy density, (where $a_m^\dagger a_m = 0$ in (eq: 1)) can thus be approximated by an integral as follows

$$\frac{2}{\Omega}\sum_k \frac{1}{2}\hbar\omega_k = \frac{2}{8\pi^3}\int_0^\infty \frac{1}{2}\hbar ck\, 4\pi k^2 dk. \tag{6}$$

The integral (eq: 6) is obviously divergent if we let $k\to\infty$, which necessitates introducing a maximum frequency $k_c = 2\pi/\lambda_c$. Thus, the integral, evaluated for all $k<k_c$ is

$$\hbar ck_c^4/8\pi^2. \tag{7}$$

The cutoff wavelength λ_c is fairly arbitrary, but to demonstrate how quickly (eq: 7) increases, we show a few examples:

$$\begin{aligned}\text{(for comparison) 100 W lightbulb at 1 m} &\rightarrow 2.7\times10^{-8}\,J/m^3,\\ \lambda_c = 1mm &\rightarrow 6.24\times10^{-13}\,J/m^3,\\ \lambda_c = 300nm\,\text{(near UV)} &\rightarrow 77.1\,J/m^3,\\ \lambda_c = 10nm\,\text{(19 eV)} &\rightarrow 6.24\times10^{7}\,J/m^3,\\ \lambda_c = 1.6\times10^{-35}m\text{ (the Planck length)} &\rightarrow 9.5\times10^{114}\,J/m^3.\end{aligned}$$

At long wavelengths, this energy is fairly negligible, but once the visible region has been included, it rapidly blows up. However, since this is the minimum energy of the field, we cannot extract any useful work from it so it does not necessarily create a paradox. The dark energy referred to in (eq: 2) is also much smaller than any of the above energy densities. However, the zero-point energy can produce a noticeable effect between two close conductors.

3 Casimir force

Consider a box of dimensions $L\times L\times L$. A conducting plate is inserted at a distance R from one of the faces ($R \ll L$). This alters the boundary conditions causing a frequency shift of the zero-point energy in the box. The change in the zero-point energy is

$$\Delta W = W_R + W_{L-R} - W_L, \tag{8}$$

where W_R is the energy on the left side of the plate (from 0 to R), W_{L-R} is the energy on the right side of the plate (from R to L) and W_L is the energy of the box before the plate was inserted.

To find the energy of each region, we multiply the integral (eq: 6) by the volume of the region. Let $k=\sqrt{k_x^2+k_y^2+k_z^2}$ so the integrals are

$$\begin{aligned}W_L &= \frac{2L^3}{\pi^3}\int\int\int_0^\infty \frac{1}{2}\hbar ck\, dk_x dk_y dk_z,\\ W_{L-R} &= \frac{2L^2(L-R)}{\pi^3}\int\int\int_0^\infty \frac{1}{2}\hbar ck\, dk_x dk_y dk_z,\\ W_R &= \sum_{n=0}^\infty \theta_n \frac{L^2}{\pi^2}\int\int_0^\infty \frac{1}{2}\hbar ck\, dk_y dk_z.\end{aligned}$$

For W_R we take $k_x = n\pi/R$ and $\theta_n = \{ \begin{array}{l} 1 \text{ for n=0} \\ 2 \text{ for n>0} \end{array}$. θ_n accounts for the two polarization states when n>0.

3.1 A short riff on the cutoff function.

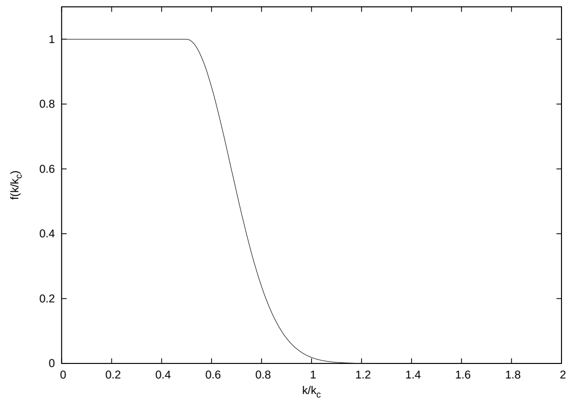

Figure 1: An example cutoff function.

We need to restrict the values of k in the above integrals, since they are, again, divergent. The high frequency modes are not noticed by the conducting plate. Thus, we can effectively ignore the high k modes. To do this, we introduce a cutoff function,

$$\begin{aligned} f(k/k_c) &\rightarrow 1 \text{ for } k \ll k_c, \\ f(k/k_c) &\rightarrow 0 \text{ for } k \gg k_c. \end{aligned}$$

An example cutoff function is plotted in (fig. 1).

3.2 Evaluating the energy shift

Applying the cutoff function to (eq: 8), we find

$$\Delta W = \hbar c \frac{L^2}{\pi^2} \left(\sum_{n=0}^{\infty} \theta_n g\left(\frac{n\pi}{R}\right) - \frac{R}{\pi} \int_0^{\infty} g(k_x) dk_x \right) \tag{9}$$

where

$$g(k_x) = \int \int_0^{\infty} k\, f(k/k_c) dk_y dk_z, \tag{10}$$

and k and $k_x$are as defined above. Through a set of substitutions (Ballentine p.537), we arrive at a simplified expression for (eq: 9) involving $\omega = n_1^2 + n_2^2 + n_3^2$,

$$\Delta W = \frac{\hbar c L^2 \pi^4}{4\pi^2 R^3} \left(\sum_{n=0}^{\infty} \theta_n F(n) - \int_0^{\infty} F(n) dn \right) \tag{11}$$

where

$$F(n) = \int_{n^2}^{\infty} \sqrt{\omega} f\left(\frac{\pi\sqrt{\omega}}{R\, k_c}\right) d\omega.$$

If we evaluate the difference between the sum and the integral by means of the Euler-Maclaurin formula [2], we find an energy shift of

$$\Delta W = -\hbar c \frac{\pi^2}{720} \frac{L^2}{R^3}.$$

This energy difference will produce a force

$$F = -\frac{\partial \Delta W}{\partial R} = -\hbar c \frac{\pi^2}{240} \frac{L^2}{R^4}.$$

Alternately, one can look at this as a pressure on the plate (of area $A = L^2$) of

$$P = \frac{F}{A} = -\hbar c \frac{\pi^2}{240 R^4}.$$

4 Experimental verification

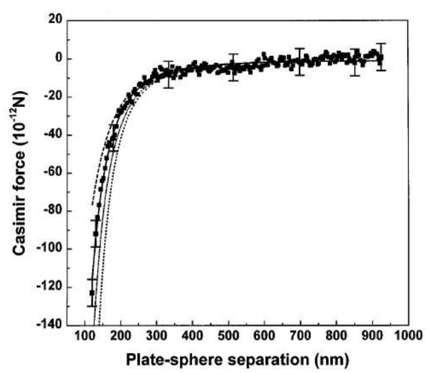

Figure 2: The measured average Casimir force is shown as square dots. The solid line is the theoretical Casimir force including all corrections. (from [3], fig 4)

Experiments have been carried out to look for the existence of this force. The first by Sparnaay in 1958 is cited in the textbook. More recently, Mohideen and Roy [3] and Lamoreaux [4], among others, have measured this effect to sub-micron scales, with an accuracy of a few percent (see fig: 2). The experiments they performed involve measuring the force on an uncharged, conducting sphere or plate when it is brought close to another uncharged, conducting plate. For a long list of examples, see [5] and a summary of recent (pre-2004) results [6].

References

[1] Casimir, H. B. G. "On the Attraction Between Two Perfectly Conducting Plates." Proc. Kgl. Ned. Akad. Wet. 60, 793, 1948.

[2] http://mathworld.wolfram.com/Euler-MaclaurinIntegrationFormulas.html

[3] U. Mohideen and A. Roy, Phys. Rev. Lett., 81, p. 4549 (1998).

[4] Lamoreaux, Phys. Rev. Lett. 78, 5-8 (1997).

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[6] K. A. Milton, J.Phys. A37 (2004) R209