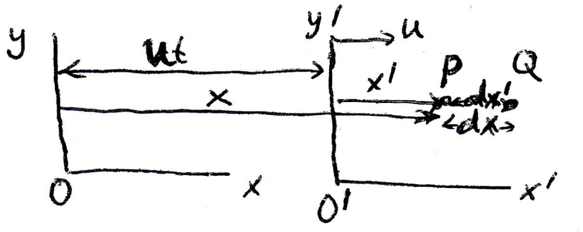
Chapter 37 (contd.)

Special Theory of Relativity

The Lorentz Coordinate Transformation



We have seen that the classical Galilean transformation equations between coordinates in s and s' are

$$x' = x - ut$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

We know that according to these transformation equations, the separation between two events P and Q

$$dx' = dx$$
$$dt' = dt$$

These equations do not give us the time dilation and length contraction that is supposed to happen when two coordinates move with respect to each other. This means that these transformation equations need modification.

Furthermore, since according to this transformation

$$u_x' = u_x - u, \quad u_y' = u_y, \quad u_z' = u_z$$

this transformation is not consistent with Einstein's postulate that velocity of light in vacuum does not depend on the motion of the coordinate system, because this will give us $c' = c - u$ and not c, a result which is inconsistent with Einstein's postulate and experiment (Michelson-Morley).

Thus the classical transformation equations must be modified to be consistent with Einstein's postulate, but they should reduce to classical results for small u. The correct relativistic transformation equations are known as Lorentz transformation.

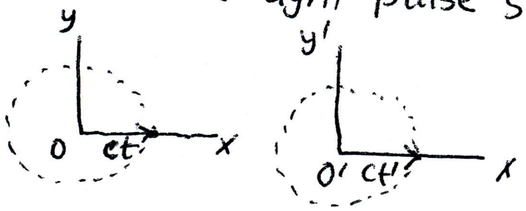
Let us assume that the transformation equation is modified as

$$x' = K(x - ut),$$ where K is a constant which can depend on u and c but not on the coordinates.

The inverse transform must look the same except for the sign of u, i.e.

$$x = K(x' + ut')$$

Let us assume that the two coordinates were coincident at $t=0$ and a light pulse starts from the origin O and O' at $t = t' = 0$



The equation for the wave front for the light pulse is

$x = ct$ in frame S

$x' = ct'$ in frame S'.

Substituting these in the transformation equations, we have

$$ct' = K(ct - ut) = K(c-u)t$$

$$ct = K(ct' + ut') = K(c+u)t'$$

Multiplying the two equations

$$c^2 t t' = K^2 (c^2 - u^2) t t'$$

$$\Rightarrow \quad K^2 = \frac{c^2}{c^2 - u^2} = \frac{1}{1 - u^2/c^2} \Rightarrow \boxed{K = \frac{1}{\sqrt{1 - u^2/c^2}} = \gamma}$$

Thus the transformation equs are

$$\left.\begin{aligned} x &= \gamma(x' + ut') \\ x' &= \gamma(x - ut) \end{aligned}\right\} \text{ combining } \quad x = \gamma\left[\gamma(x - ut) + ut'\right]$$

Solving for t' in terms of x and t, we get

$$t' = \gamma\left(t - \frac{ux}{c^2}\right)$$

Thus the times are not identical in the two coordinate systems

For motion of the primed system in x-direction, the complete transformation equations are

$$\begin{aligned} x' &= \gamma(x - ut) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{ux}{c^2}\right) \end{aligned}$$

The inverse transformation:

$$\begin{aligned} x &= \gamma(x' + ut') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{ux'}{c^2}\right) \end{aligned}$$

We see that the value for t' assigned to an event by observer O' depends on both time t and on the coordinate x as measured by observer O. Therefore in special relativity, time and space are not separate concepts but are interwoven with each other in what we call space-time. In Galilean transformation, of course, $t' = t$.

When $u \ll c$, the Lorentz transformation reduces to Galilean transformation. Here $\gamma \to 1$, and $\frac{u}{c} \to 0$.

Time dilation and length contraction can be easily derived from the Lorentz transformation.

Consider two events at the same place x' at times t_1' and t_2' in S'. What is the time interval between the two events in S?

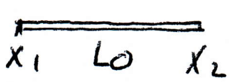
$$t_1 = \gamma\left(t_1' + \frac{u x'}{c^2}\right), \quad t_2 = \gamma\left(t_2' + \frac{u x'}{c^2}\right)$$

Subtracting 1st eqn. from the 2nd

$$t_2 - t_1 = \gamma\,(t_2' - t_1') \Rightarrow \Delta t = \gamma \Delta t_0 . \ \Delta t_0 = t_2' - t_1' \text{ is}$$

the proper time interval.

Length contraction: Meterstick fixed on the ground



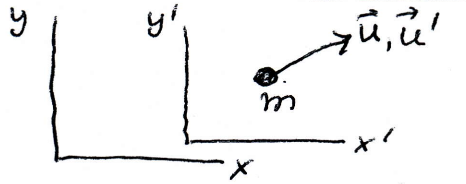
$$L_0 = x_2 - x_1 = u(t_2 - t_1)$$

Length measured in the moving primed coordinate system

$$L' = u\,(t_2' - t_1') = \frac{u\,(t_2 - t_1)}{\gamma}$$

$$\Rightarrow \frac{L'}{L_0} = \frac{1}{\gamma} \Rightarrow L' = \frac{L}{\gamma}$$ $\Rightarrow$ Measured length in the moving coordinate system is lower.

The Lorentz Velocity Transformation



If u and u' are the velocities of a mass m measured in the two coordinate system, we can find a relationship between them. If the particle moves a distance dx and dx' in time dt and dt', in the two coordinate systems,

$$dx' = \gamma\,(dx - u\,dt)$$
$$dt' = \gamma\left(dt - \frac{u\,dx}{c^2}\right)$$

So $u_x' = \dfrac{dx'}{dt'} = \dfrac{dx - u\,dt}{dt - \frac{u}{c^2}dx} = \dfrac{v_x - u}{1 - \frac{u v_x}{c^2}}$ where $v_x = \dfrac{dx}{dt}$.

Similarly, we can show that

$$v_y' = \frac{v_y}{\gamma\left(1 - \frac{u v_x}{c^2}\right)} \quad \text{and} \quad v_z' = \frac{v_z}{\gamma\left(1 - \frac{u v_x}{c^2}\right)}.$$

Note when $u \to 0 \Rightarrow \boxed{v_x' = v_x - u,\ v_y' = v_y \text{ and } v_z' = v_z}$ since $\frac{u v_x}{c^2} \to 0$ and $\gamma \to 1$

So the Galilean transformation is satisfied for small values of u.

Example: Speed of light in different coordinate systems

Light pulse moves along the x-axis in S with speed $v_x = c$. What is the speed of the pulse in the primed system S'?

In the primed system speed of light is

$$v_x' = \frac{c - u}{1 - \frac{cu}{c^2}} = \frac{c(1 - u/c)}{(1 - u/c)} = c$$

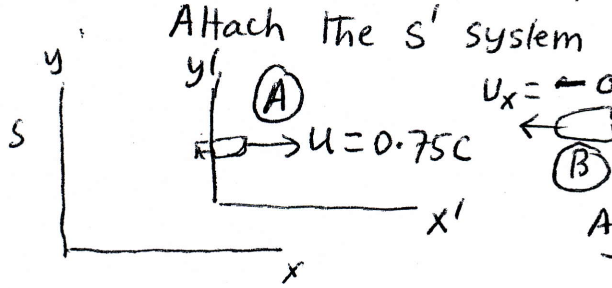
So if light moves with speed c in S, it will move with speed c in S' also, independent of relative motion of S and S' $\Rightarrow$ consistent with Einstein's 2nd postulate.

Inverse velocity transformation: $v_x = \dfrac{v_x' + u}{1 + u v_x'/c^2}$

Example: Relative velocity of two spacecrafts

Two spacecrafts moving toward each other with speeds 0.75c and 0.85c with respect to ground. Find the velocity of one (say, the 2nd one) with respect to the other (the 1st one).

Attach the S' system with the first one $\Rightarrow u = 0.75c$. The speed of 2nd one w.r.t. ground $v_x = -0.85c$. Need to find v_x'.



According to Galilean transformation

$$v_x' = v_x - u = -0.85c - 0.75c = -1.6c$$

Speed of B as seen by A will be greater than c which is impossible.

According to Lorentz transformation

$$u_x' = \frac{v_x - u}{1 - v_x u/c^2} = \frac{-0.85c - 0.75}{1 - (-0.85)(0.75c)/c^2} = -0.98c < c, \text{ OK}$$

Relativistic Momentum

The special theory of relativity demands that all laws of physics must remain invariant (unchanged) under Lorentz transformation. Although the laws of electricity and magnetism conform to Lorentz transformation, unfortunately it is not the case for laws of mechanics. So we must generalize the definitions of momentum and energy and change the laws of mechanics so that they conform to Lorentz transformation. Of course, in the limit of $u \ll c$ and $v \ll c$, they must result to classical nonrelativistic results.

For example, conservation of momentum is a general conservation law of physics. If momentum is conserved in a collision between two particles in one coordinate system, it should be conserved if calulated in a moving coordinate system. If we use the classical definition $\vec{p} = m\vec{v}$, the momentum is not conserved in a moving coordinate system. So the definition of $\vec{p}$ needs to be changed. The relativistic momentum which preserves its conservation in collisions is defined as

$$\boxed{\vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}} = \gamma m\vec{v}} \quad \text{where } \gamma = \frac{1}{\sqrt{1-v^2/c^2}}.$$

(Note that this γ is different from the $\gamma = \frac{1}{\sqrt{1-u^2/c^2}}$ referred to the motion of the coordinate system).

In the limit $v \ll c$, $\gamma \to 1$ and the relativistic $\vec{p}$ approaches the classical expression $\vec{p} = m\vec{v}$

Newton's Second Law

The relativistic force $\vec{F}$ acting on a particle of momentum $\vec{p}$ can then be obtained from the most general form of the second law

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$$

When the net force and velocity are both along the x-axis, we have

$$\frac{dp}{dt} = m\frac{d}{dt}\frac{u}{\sqrt{1-u^2/c^2}} = \frac{m\frac{du}{dt}}{\sqrt{1-u^2/c^2}} + mu\left(-\frac{1}{2}\right)\frac{\left(-\frac{2u}{c^2}\right)\frac{du}{dt}}{(1-u^2/c^2)^{3/2}}$$

$$= m\frac{du}{dt}\frac{1}{(1-u^2/c^2)^{3/2}}$$

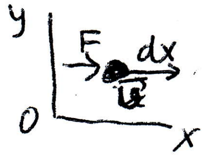
$$\Rightarrow F = \frac{m}{(1-u^2/c^2)^{3/2}}a$$ where $a = \frac{du}{dt}$ is the acceleration along the x-axis.

Solving for $\boxed{a = \frac{F}{m}\left(1-\frac{u^2}{c^2}\right)^{3/2}.}$

This shows that as the particle's speed approaches c, the acceleration under a constant force decreases to zero. Hence it is impossible to accelerate a particle from rest to a speed $u \geq c$. This argument implies that the speed of light is the ultimate speed. No particle of positive mass can attain speed greater than the speed of light.

Relativistic Work and Kinetic Energy

Just like the momentum, the relativistic kinetic energy of a particle takes a different form which, of course, reduces to the non-relativistic value at small values of v. We can derive the form of the kinetic energy by using the work-kinetic energy theorem, i.e., the change in kinetic energy of a particle is equal to the work done by a force on the particle. From the definition of work



$$dW = F\,dx\,, \qquad dx = v\,dt$$

$$W = \int_{x_1}^{x_2} F\,dx = \int \frac{dp}{dt}\,dx = \int \frac{m\,\frac{dv}{dt}}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}\,v\,dt$$

$$W = m\int_0^v \frac{v\,dv}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}.$$

This integral can be done in the closed form and we get

$$W = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2$$

By the work-energy theorem, the kinetic energy of the particle K is

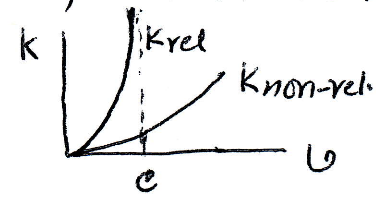
$$K - 0 = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2$$

The relativistic kinetic energy is

$$\boxed{K = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2 = (\gamma-1)mc^2}$$

For small v, $\dfrac{1}{\sqrt{1-v^2/c^2}} = \left(1-\frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\left(\frac{v^2}{c^2}\right)^2 + \cdots$

$K \simeq mc^2\left(1+\frac{v^2}{2c^2}\right) - mc^2 = \frac{1}{2}mv^2$, which is the non-relativistic form of the kinetic energy.



The constant mc^2 which is independent of velocity is called the rest energy $\boxed{E_R = mc^2}$ This eqn shows mass is a form of energy where c^2 is the conversion factor. This shows that a small mass is a source of enormous amount of energy — a concept of fundamental importance in nuclear physics

The total energy of the particle is

$$\boxed{E = K + mc^2 = \gamma mc^2}$$

In many situation one would want to express energy in terms of momentum rather than v, This is accomplished by combining

$$E = \gamma mc^2 \quad \text{and} \quad \vec{p} = \gamma m\vec{v}$$

Taking square $\quad E^2 = \gamma^2 m^2 c^4$

$$p^2c^2 = \gamma^2 m^2 c^2 v^2$$

Subtracting

$$E^2 - p^2c^2 = \gamma^2 m^2 c^2 (c^2 - v^2) = \frac{m^2c^2(c^2-v^2)}{1-v^2/c^2} = m^2c^4$$

$$E = \sqrt{m^2c^4 + p^2c^2} = c\sqrt{p^2 + m^2c^2}$$

Again we see that for a particle at rest ($p=0$)

$$E = mc^2$$

Above equation for E also suggests that a particle may have energy and momentum even when it has no rest mass. If $m=0$ $E = pc$ (zero rest mass). Zero rest mass particles do exist. Such particles always travel at the speed of light in vacuum. The prime example is the photon, the quantum of electromagnet. radiation. We will discuss this later.

The speed of a relativistic particle

$$\frac{E}{c^2} = \gamma m = \frac{\vec{p}}{\vec{v}} \Rightarrow \boxed{\vec{v} = \frac{\vec{p}c^2}{E}}$$

1. Examples: Calulate

a) The rest energy of an electron

$$m = 9.11 \times 10^{-31}\,kg$$

$$E_R = mc^2 = (9.11 \times 10^{-31})(3\times10^8)^2 = 8.20 \times 10^{-14}\,J$$

Better to express in electron-volts. Remembering $1\,eV = 1.6\times10^{-19}\,J$

$$E_R = mc^2 = \frac{8.20\times10^{-14}\,J}{1.6\times10^{-19}\,J/eV} = 0.511\times10^{6}\,eV = 0.511\,MeV$$

$\Rightarrow$ mass of an electron $m = 0.511\,MeV/c^2$

b) Relativistic momentum of an electron moving with speed $v = 0.8c$

$$p = \gamma m v \cdot \text{ where } \gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-0.64}} = \frac{1}{0.6} = 1.67$$

$$\Rightarrow p = (1.67)\left(0.511\,\frac{MeV}{c^2}\right)(0.8c) = 0.683\,\frac{MeV}{c}$$

$\frac{MeV}{c}$ is a convenient unit of momentum

c) Total energy of the electron

$$E = \gamma mc^2 = (1.67)(0.511\,MeV) = 0.853\,MeV$$

d) Kinetic energy of the electron

$$K = E - mc^2 = 0.853\,MeV - 0.511\,MeV = 0.342\,MeV.$$

2. An electron has a total energy 5 times its rest energy. What are its momentum and speed?

$$p^2c^2 = E^2 - (mc^2)^2 = (5mc^2)^2 - (mc^2)^2 = 24(mc^2)^2$$

$$pc = \sqrt{24}\,mc^2 = (4.90)(0.511\,MeV) = 2.50\,MeV$$

$$p = 2.50\,\frac{MeV}{c}$$

Speed can be found from $v = \frac{pc^2}{E}$

$$= \frac{(\sqrt{24}\,mc^2)\,c}{5mc^2} = 0.98c$$

Mass and Energy

The expression for the total energy of a particle $E = \gamma mc^2$ shows that the particle has enormous energy even when it is ~~rest~~ rest i.e., $\gamma = 1$. This equation shows the equivalence of mass and energy. The atomic and nuclear bombs were created on the basis of this equivalence. Nuclear power plants also utilize this principle to create usable power.

In a nuclear reactor, uranium nucleus undergoes fission, upon bombardment of neutrons, producing several lighter nuclei

$$n + {}^{235}U \rightarrow ({}^{236}U)^* \rightarrow {}^{141}_{56}Ba + {}^{92}_{36}Kr + 3n + Q$$

Q = Rest energy of ${}^{235}U$ − rest energy of (Ba + Kr + 2n)
= 175 MeV

which appears as the Kinetic energy of the nuclear fragments and this energy is then dissipated in water raising its internal energy. The internal energy produces steam for generation of electric power.

In nuclear fusion, two lighter nuclei can fuse to produce a heavier nucleus such as

$$^2H + {}^3H \rightarrow {}^4He + n + 17.6\ MeV$$

or $$^2H + {}^2H \rightarrow {}^4He + \text{energy}$$

Since the rest energy of ^2H and ^3H is higher than that ^4He & neutron, the difference shows up as available energy. However, overcoming the Coulomb repulsion for nuclear fusion to take place, has been a big problem in carrying out nuclear fusion in the controlled way. It will be a source of very clean energy of enormous amount if this can be achieved.