Energy in Wave Motion

There is energy associated every wave motion. we receive energy from the sun through the electromagnetic wave that it radiates. When we generate transverse wave on a string we supply energy which propagates through the medium. To fix our ideas let us consider transfer of energy by sinusoidal wave on strings

Rate of Energy Transfer by Sinusoidal Waves on Strings

As the string oscillates, there is kinetic energy associated with the speed of the particles and there is also potential energy associated with displacement from equilibrium.

We can argue like the simple harmonic oscillator, the total energy per unit length will be

$$\frac{E}{\text{length}} = \frac{K}{\text{length}} + \frac{U}{\text{Length}}$$

Also

$$\frac{E}{\text{length}} = \frac{K_{max}}{\text{length}}$$

$$\frac{K}{\text{length}} = \frac{1}{2}\mu v_y^2 = \frac{1}{2}\mu \omega^2 A^2 \sin^2(kx-\omega t)$$

Thus

$$E/\text{length} = \frac{1}{2}\mu\omega^2 A^2$$

The amount of energy generated in one wavelength, λ

$$E_\lambda = \frac{E}{\text{length}} \times \lambda = \frac{1}{2}\mu\omega^2 A^2 \lambda$$

As the wave moves along the string, this amount of energy passes by a given point on the string during one period of oscillation. Therefore, the power or rate of energy transfer associated with the wave

$$\overline{P} = \frac{E_\lambda}{T} = \frac{1}{2}\mu\omega^2 A^2 \frac{\lambda}{T} = \frac{1}{2}\mu\omega^2 A^2 v = \frac{1}{2}\mu\omega^2 A^2 \sqrt{\frac{F}{\mu}}$$

$$= \frac{1}{2}\sqrt{\mu F}\,\omega^2 A^2$$

Substituting for $\omega = v k$, we can write

$$\bar{P} = \frac{1}{2} \mu v^3 A^2 k^2$$

Short wavelengths carry more energy.

Above is the general result for all mechanical waves: $\bar{P}$ is proportional to A^2 and ω^2 whereas electromagnetic wave has $\bar{P} \propto A^2$ but independent of ω.

Example: <u>Power Supplied to a Vibrating String</u>

Linear mass density of string $\mu = 5 \times 10^{-2}$ kg/m and tension on string $F = 80$ N. How much power needed to be supplied to generate sinusoidal waves at frequency $f = 60$ Hz and an amplitude of 6 cm.

$$v = \sqrt{\frac{F}{\mu}} = \left(\frac{80}{5 \times 10^{-2}}\right)^{1/2} = 40.0 \text{ m/s}$$

$$f = 60 \Rightarrow \omega = 2\pi f = 2\pi(60) = 377/s$$

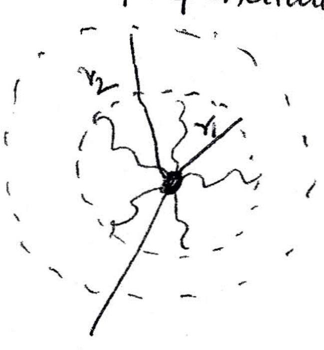
$$A = 6 \times 10^{-2} m$$

$$\bar{P} = \frac{1}{2} \mu \omega^2 A^2 v = \frac{1}{2}(5 \times 10^{-2})(377)^2 (6 \times 10^{-2})^2 (40)$$

$$= 512 \text{ W}$$

<u>Wave Intensity</u>

Waves on a string propagate energy/power in one dimension. Other kinds of waves like sound waves, seismic waves can carry energy in all three dimensions. For these waves, we fine the intensity of the wave at any point is the <u>time average rate at which energy is transported by the wave, per unit area perpendicular to the direction of propagation of the wave</u>



Source of wave

If the power output of the source is $\bar{P}$, then the intensity at any point on the spherical surface of radius r_1 is

$$I_1 = \frac{\bar{P}}{4\pi r_1^2} \quad \text{(inversely proportional to } r^2\text{)}$$

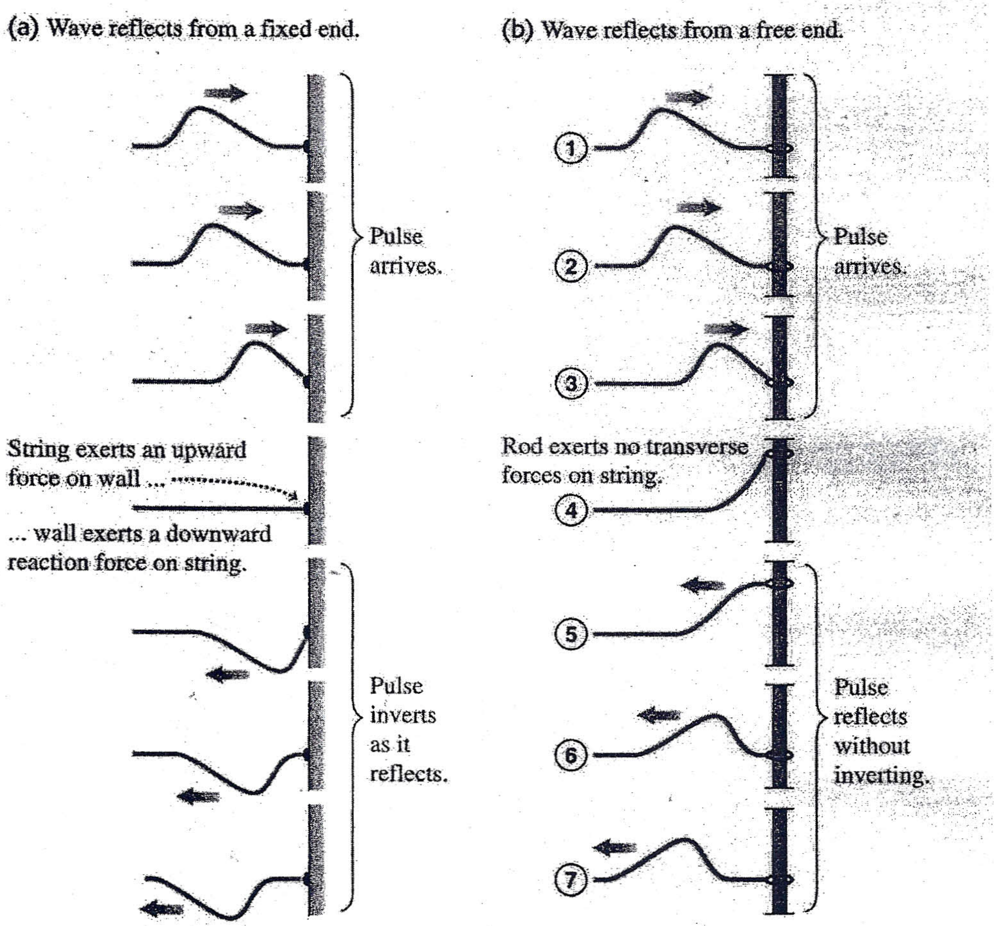
Since the same power propagates to the spherical surface of radius r_2, $I_2 = \frac{\bar{P}}{4\pi r_2^2} \Rightarrow \frac{I_1}{I_2} = \frac{r_2^2}{r_1^2}$

The inverse square law of intensity.

Boundary Conditions, Wave Interference and Superposition

When a traveling wave strikes the boundary, all or part of it is reflected (echo for sound wave). If the transverse wave or pulse on a string hits a fixed end, the reflected wave gets inverted as shown on the left below. If the wave gets reflected from a free end, the pulse reflects without inverting as shown on the right below. These conditions at the end of a string are called the boundary conditions.

84. Figure 15.19 Reflection of a wave pulse from a fixed and free end



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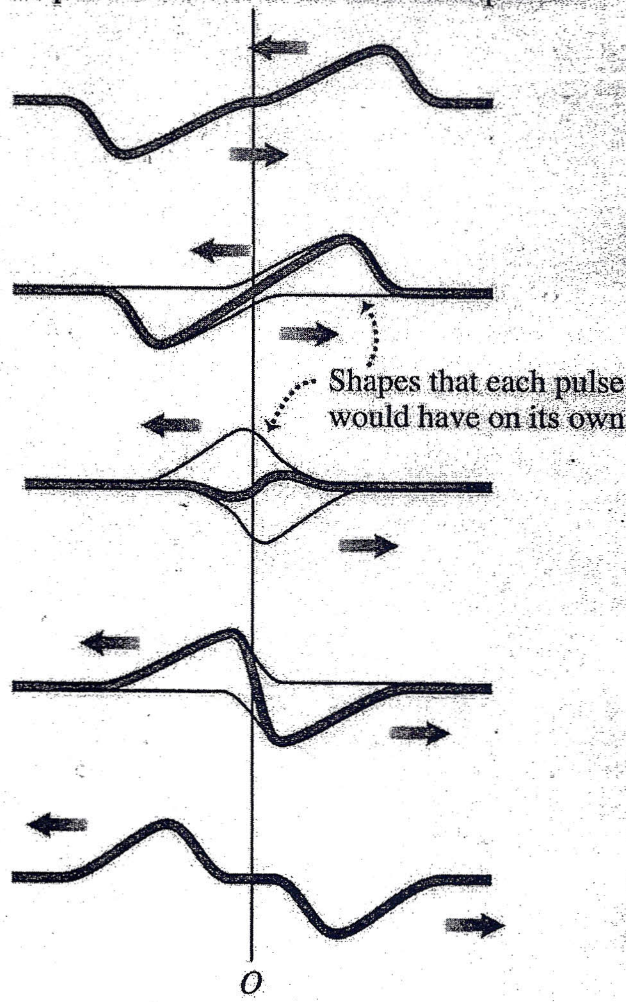
Interference

When two pulses on a string move in opposite direction as shown below, the total displacement at any point is the algebraic sum of the displacements of the individual pulses. Figure 15.20 shows the total displacements of two pulses of opposite sign. In the region of overlap, they cancel each other or undergo destructive interference as shown below.

85. Figure 15.20 Interference of two wave pulses (one upright, one inverted) traveling in opposite directions

As the pulses overlap, the displacement of the string at any point is the algebraic sum of the displacements due to the individual pulses.



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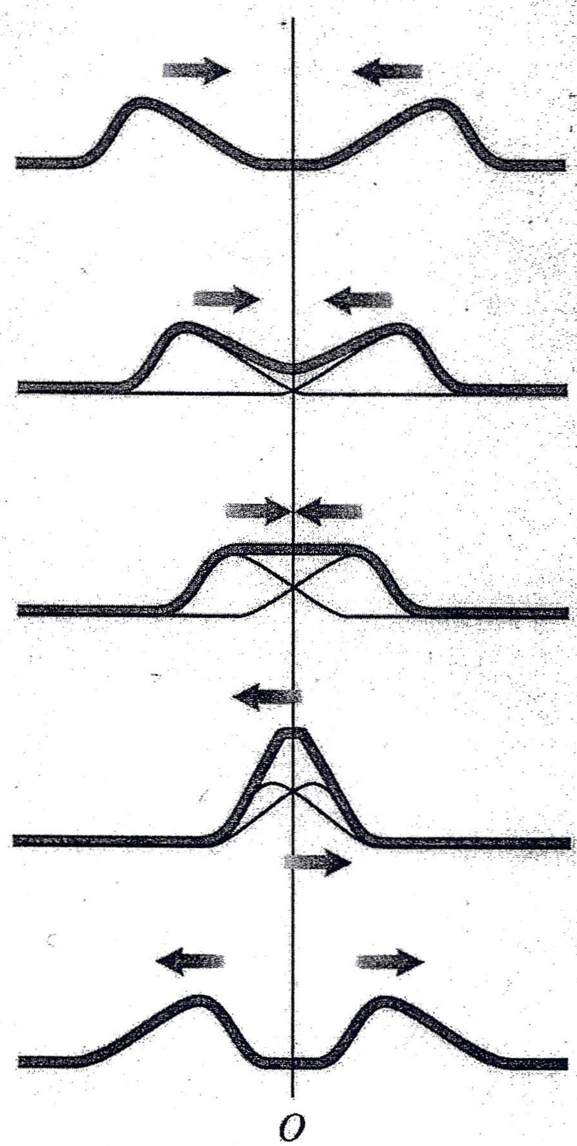
When two positive pulses move in opposite directions on a string, in the region of overlap they add up to give a higher displacement than the individual pulses. We say that there is constructive interference.

86. Figure 15.21 Interference of two wave pulses (both upright) traveling in opposite directions



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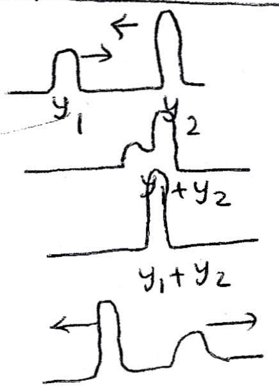
The Principle of Superposition

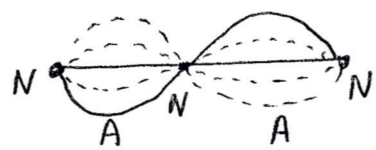
When two or more waves move through a medium, the resultant displacement of any point is the algebraic sum of displacements due to individual waves. This is known as the superposition principle. (Result of the linearity of the wave equation).



Standing Waves on a String

When a sinusoidal wave is reflected from a free or fixed end of a stretched string, the incident and the reflected wave combine according to the principle of superposition and the resultant wave is a standing wave on the string.

The snapshot of one of the standing waves on a string will be as shown below:

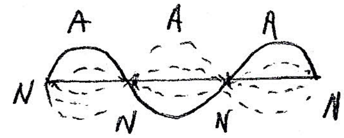


Characteristics: ① Some points on the string is always at rest and they are called nodes.

② Some points have the largest amplitude of oscillation. They are called the antinodes.

Note that two successive nodes are separated by an antinode. Similarly there is a node between two successive antinodes. Since the wave pattern does not seem to move in either direction along the string, it is called a standing wave

Can check that nodes and antinodes occur $\frac{\lambda}{2}$ apart and between two nodes there is an antinode.



Separation between successive node and antinode is $\frac{\lambda}{4}$.

Energy is not transported by standing wave. Since node points are always fixed, no energy can be transported accross nodes.

Example: $y_1 = (4\text{ cm}) \cos(3x - \omega t)$

$y_2 = (-4\text{ cm}) \cos(3x + \omega t)$, x and y are measured in cms.

$A = 4\text{ cm}$, $k = 3\text{ rad/cm}$, $\omega = 2\text{ rad/s}$

(a) Wave form

$$y(x,t) = 2A \sin kx \sin \omega t = (8\text{ cm}) \sin 3x \sin 2t$$

(b) Maximum displacement at $x = 2.3$ cm

$$A' = (8\text{ cm}) \sin 3x \Big|_{x = 2.3\text{ cm}} = 8\text{ cm} \sin(6.9\text{ rad})$$

c) Positions of antinodes + nodes; $k = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{k} = \frac{2\pi}{3}$ cm

$x_{anti} = \left(\frac{2n+1}{4}\right)\lambda = \left(\frac{2n+1}{4}\right)\left(\frac{2\pi}{3}\right) = (2n+1)\frac{\pi}{6}$, $n = 0, 1, 2..$; $x_{node} = n\frac{\lambda}{2} = n\left(\frac{\pi}{3}\right)$ cm, $n = 1, 2..$

Normal Modes on a Stretched String

For above description we assumed that wave is reflected from one end. The allowed frequencies or wavelengths of vibrations of a string fixed at both ends, i.e.

$$y(x=0,t) = 0 \quad \text{and} \quad y(x=L,t) = 0$$

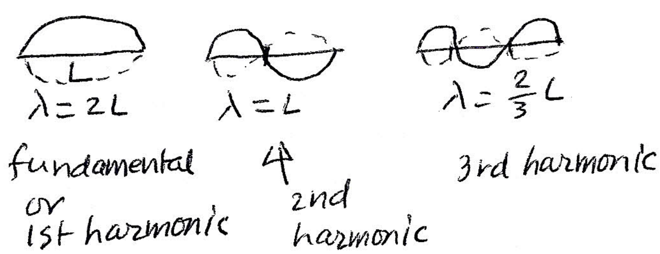
The second boundary condition means

$$2A \sin kL \sin \omega t = 0 \quad \text{at any } t.$$

$$\Rightarrow \quad \sin kL = 0 \Rightarrow kL = 0, \pi, 2\pi \cdots = n\pi, n = 1, 2 \cdots$$

$$\Rightarrow \frac{2\pi}{\lambda} L = n\pi \Rightarrow \lambda_n = \frac{2L}{n}$$ [resonance or standing wave condition]

$$\Rightarrow \lambda = 2L, L, \tfrac{2}{3}L, \tfrac{1}{2}L \cdots$$



Allowed Frequencies

Since $f\lambda = v$

$$f_n = \frac{v}{\lambda_n} = \frac{vn}{2L} = \frac{n}{2L}\sqrt{\frac{F}{\mu}}, \quad n = 1, 2, 3 \ldots$$

These are the frequencies allowed on a string of length L and are called the normal modes.

The fundamental or the 1st harmonic

$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$$

All other harmonics are $f_n = n f_1$

Higher harmonics along with the fundamental constitute a harmonic series.

These are the frequencies that are generated in a string musical instrument such as a guitar, violin, piano, etc. These vibrations are the sources of sound that a listener hears.

Example: A middle C on a piano has fundamental frequency of 262 Hz and the A note has a fundamental frequency 440 Hz.

(A) Next two harmonics of C string

$$f_2 = 2f_1 = 524 \text{ Hz}, \quad f_3 = 3f_1 = 786 \text{ Hz}$$

[B] Ratio of the tensions in two strings if $\mu_c = \mu_A$ i.e. mass/length is the same

$$f_{1A} = \frac{1}{2L}\sqrt{\frac{F_A}{\mu}}, \quad f_{1c} = \frac{1}{2L}\sqrt{\frac{F_c}{\mu}}$$

$$\frac{F_A}{F_c} = \left(\frac{f_{1A}}{f_{1c}}\right)^2 = \left(\frac{440}{262}\right)^2 = 2.82$$

Wave form of The Standing Wave

Consider the superposition of the incident wave moving to the right and the reflected wave from a fixed end moving to the left:

$$y_1(x,t) = A\cos(kx-\omega t)$$
$$y_2(x,t) = -A\cos(kx+\omega t)$$

Resultant wave

$$y(x,t) = y_1+y_2 = A\left[\cos(kx-\omega t) - \cos(kx+\omega t)\right]$$

$$= A\left[(\cos kx\cos\omega t + \sin kx\sin\omega t) - (\cos kx\cos\omega t - \sin kx\sin\omega t)\right]$$

$$= [2A\sin kx]\sin\omega t$$

This is the equation of a standing wave. This does not have the characteristic of a traveling wave which would be a function of $(kx \pm \omega t)$. The standing wave looks more like a simple harmonic motion than a traveling wave. Its characteristics are

1. A particle at a given x undergoes SHM of frequency ω with respect to time
2. The amplitude depends on x: $A' = 2A\sin kx$

Thus maximum amplitude $A' = 2A$ occur at points where $\sin kx = 1$ and amplitude is zero at points where $\sin kx = 0$.

Antinodes; The points at which maximum amplitude occurs are called antinodes. They occur at

$$\sin kx = 1 \Rightarrow kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \qquad = \frac{2\pi}{\lambda}x$$

Since $k = \frac{2\pi}{\lambda} \Rightarrow x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \ldots = \frac{(2n+1)\lambda}{4}$, $n = 0, 1, 2\ldots$

Note adjacent antinodes are $\frac{\lambda}{2}$ apart.

Nodes: The points at which $A' = 0$ are called nodes. No oscillation at nodes

$$\sin kx = 0 \Rightarrow kx = 0, \pi, 2\pi, 3\pi, \ldots \qquad = \frac{2\pi}{\lambda}x$$

$$\Rightarrow x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2} \ldots = \frac{n\lambda}{2}, \; n = 0, 1, 2\ldots$$