Chapter 15
Mechanical Waves

Our discussion of oscillators is important because it leads to the understanding of waves. Since waves appear in different forms in nature, it is important to understand their behavior.

The wave can be defined as a disturbance which propagates in time from one region of space to another.

There are different kinds of waves

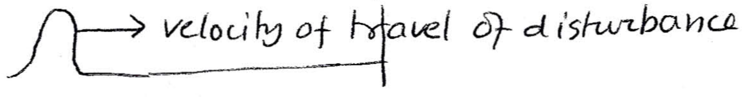
1. Mechanical waves: Waves that travel within some material called a medium. Waves on a string, sound waves, water waves, etc.
2. Electromagnetic Waves: can propagate through vacuum – visible light, radio waves, microwaves, ultraviolet, x-rays, γ-rays.
3. Quantum (matter) Waves: waves that can be associated with matter and are needed to explain some physical phenomena.

In a wave propagation the medium itself does not propagate. Every bit of the medium just oscillates about their equilibrium positions as the wave propagates.

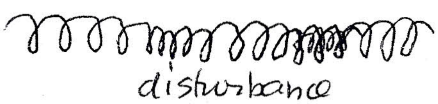
Waves can be further classified as

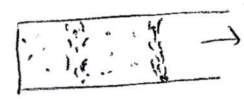
1. Transverse: Disturbance or displacement perpendicular to the direction of propagation
   (waves on a string, electromagnetic waves)

Disturbance on a string attached at one end



2. Longitudinal: Disturbance in the same direction as the direction of propagation
   (sound waves, slinky, spring)



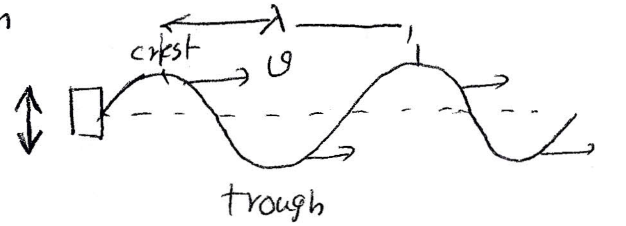


3. Mixed : Disturbance with transverse and longitudinal components (water waves)

Periodic Waves

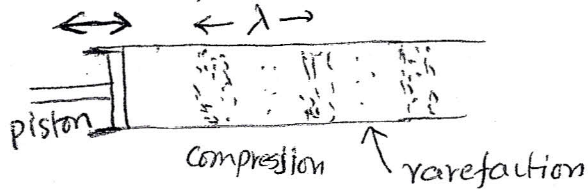
A wave (disturbance) that repeats itself. The simplest periodic wave is the harmonic wave where the medium (particles) moves in simple harmonic motion.

A transverse periodic wave can be created on a long stretched string by moving one end up and down ~~by~~ with simple harmonic motion



Snapshot of a transverse wave

Similarly longitudinal periodic wave can be generated in a long tube filled with a fluid, with a piston attached at the left end.



snapshot of a longitudinal wave

Can identify certain inherent features of periodic waves

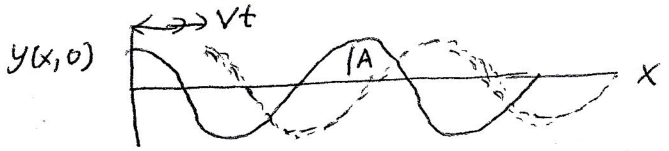
λ = wavelength, distance between two successive similar disturbance

T = time period, time taken by the wave to propage through one wavelength $\Rightarrow$ Frequency $f = \frac{1}{T}$.

Mathematical Description of a Wave

It is obvious that the displacement y of the medium will depend on x and it also true that y will depend on the time t at which the measurement is made. Thus y is a function of x and t, i.e. $y = y(x,t)$. $y(x,t)$ is called the wave function or the waveform.

Suppose we take a snapshot of a wave traveling to the right and see the following form for $y = y(x,0)$



$y(x,0)$ can be mathematically written as

$$y(x,0) = A\cos \frac{2\pi}{\lambda} x$$ where A is the amplitude

This allows $\quad y(x) = y(x+\lambda) = y(x+2\lambda)$ since $\cos x = \cos(x+2\pi n)$, $n = 0, 1, 2 \cdots$

If we take the snapshot of the wave at time t, we will get the dashed curve because every point of the wave has moved through a distance vt.

Thus disturbance at every point x at time t is then exactly the same as what it was at $x' = x - ct$ at $t = 0$

So $$y(x,t) = A\cos \frac{2\pi}{\lambda} x' = A\cos\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

There are several ways of expressing this wavefunction. Since T is the time required by the wave to travel a distance of one wavelength $\Rightarrow T = \frac{\lambda}{v} \Rightarrow v = \frac{\lambda}{T}$.

$$y(x,t) = A\cos\left[\frac{2\pi}{\lambda} x - \frac{2\pi}{\lambda}\left(\frac{\lambda}{T}\right)t\right] = A\cos 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

Defining the angular wave number k and angular frequency ω as

$k \equiv \frac{2\pi}{\lambda}$ and $\omega \equiv \frac{2\pi}{T} = 2\pi f$, we have

$$\boxed{y(x,t) = A\cos(kx - \omega t)}$$

The speed of the wave can be written in other forms

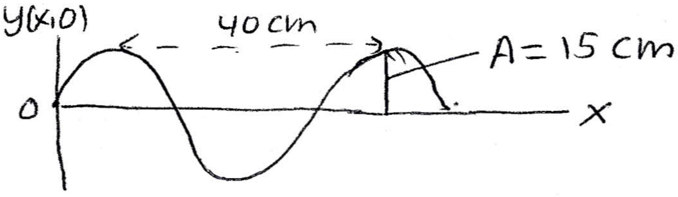
$$v = \frac{\lambda}{T} = \lambda f \quad \text{or} \quad v = \frac{2\pi\lambda}{2\pi T} = \frac{\omega}{k}$$

If the disturbance is $y(x,t)$ is not A at $x=0$ and $t=0$, the wave function will be modified to

$$y(x,t) = A\cos(kx-\omega t+\phi)$$

where ϕ (the phase factor) can be found from the initial conditions.

Example :



Find the wave function $y(x,t)$ if $\lambda = 40$ cm and $f = 8$ Hz & $A = 15$ cm

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{40\,\text{cm}} = 0.157 \text{ rad/cm}$$

$$\omega = 2\pi f = 2\pi\left(\frac{8}{s}\right) = 50.3 \text{ rad/s}$$

Initial condition $\quad y(0,0) = A\cos\phi = 0 \Rightarrow \phi = \frac{\pi}{2}$

$$y(x,t) = A\cos\left(0.157x - 50.3t + \frac{\pi}{2}\right)$$

$$= A\sin(0.157x - 50.3t),$$ x in cm and t in seconds.

Velocity and Acceleration of a point (particle) in the medium

From the expression for the wave function we can calculate the velocity and acceleration of a particle in the medium

$$y(x,t) = A\cos(kx-\omega t)$$

Velocity: $\quad v_y = \frac{\partial y(x,t)}{\partial t} = \omega A\sin(kx-\omega t)$

Acceleration $\quad a_y = \frac{\partial v_y}{\partial t} = \frac{\partial^2 y(x,t)}{\partial t} = -\omega^2 A\cos(kx-\omega t) = -\omega^2 y(x,t)$

a_y is equal to $-\omega^2$ times its displacement.

$\boxed{v_{y\,max} = \omega A}$ and $\boxed{a_{y\,max} = \omega^2 A}$

Note that v and A are out of phase.

Wave Equation

Let us calculate $\quad \frac{\partial y}{\partial x} = -kA\sin(kx-\omega t)$

$$\frac{\partial^2 y}{\partial x^2} = -k^2 A\cos(kx-\omega t)$$

$$\Rightarrow \quad \omega^2 \frac{\partial^2 y}{\partial x^2} = k^2 \frac{\partial^2 y}{\partial t^2} \quad \text{or,} \quad \frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2}$$

or,
$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$

This equation, known as the wave equation, is one of the most important equations. It indicates that a disturbance can propagate as wave along the x-axis with wave speed v.

Example: Transverse wave on a string has the form

$$y(x,t) = (0.120\,m)\sin\left[\frac{\pi x}{8} + 4\pi t\right]$$

The plus sign before $4\pi t$ indicates that the wave is propagating to the left.

Calculate A, λ, T, v, v_y and a_y at $t = 0.20\,s$ and $x = 1.6\,m$

Compare with $y(x,t) = A\cos(kx - \omega t + \phi)$

By inspection $A = 0.120\,m$, $k = \frac{\pi}{8\,m}$, $\omega = 4\pi/s$, $\phi = \frac{\pi}{2}$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\pi/8} = 16\,m \qquad T = \frac{2\pi}{\omega} = \frac{2\pi}{4\pi/s} = 0.5\,s$$

$$v = \frac{\omega}{k} = \frac{4\pi}{\pi/8} = 32\,m/s$$

$$v_y = \frac{\partial y}{\partial t} = A\omega\cos\left(\frac{\pi x}{8} + 4\pi t\right)$$
$$= (0.12)(4\pi)\cos\left[\frac{\pi}{8}(1.6) + 4\pi(0.2)\right]$$
$$= 0.48\pi\cos(0.2\pi + 0.8\pi) = -0.48\pi = -1.51\,m/s$$

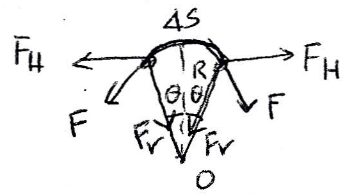
$$a_y = \frac{\partial^2 y}{\partial t^2} = -A\omega^2\sin\left(\frac{\pi x}{8} + 4\pi t\right) = -A\omega^2\sin\pi = 0$$

The Speed of Transverse Wave On Strings

The speed of transverse wave on a string is determined by the tension in the string and its mass per unit length or linear mass density. Increasing the tension would increase the restoring force and thus increase the speed. Increasing the mass would make the string more sluggish and hence decrease the speed.

Let us look at an element Δs of a vibrating string

wave on a string



$$F_r = 2T\sin\theta \simeq 2F\theta = F\frac{\Delta s}{R}$$

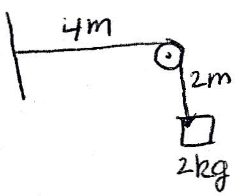
Since $\Delta s = 2\theta R$

Mass of element Δs is $m = \mu \Delta s$ where $\mu = \frac{\text{mass}}{\text{length}}$,

$$F_r = \frac{mv^2}{R}, \quad \text{the centripetal force}$$

$$\frac{F\Delta s}{R} = \frac{\mu \Delta s\, v^2}{R} \Rightarrow v^2 = \frac{F}{\mu} \Rightarrow v = \sqrt{\frac{F}{\mu}}$$

Example.



Uniform cord of length 6m and mass 0.3 kg passes over a pulley and supports a mass 2kg. What is the speed of a pulse passing ~~over the~~ along the string?

$$\left.\begin{aligned} F &= mg = 2(9.8) = 19.6\,N \\ \mu &= \frac{m}{l} = \frac{0.3}{6} = 0.05\,kg/m \end{aligned}\right\} \quad v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{19.6}{0.05}} = 19.8\,m/s$$