Energy in Simple Harmonic Motion

Let us consider the energy associated with simple harmonic motion.

Since $F_x = -kx = -\frac{\partial U}{\partial x} \Rightarrow U = \frac{1}{2}kx^2$

This is the potential energy associated with a compressed or elongated spring. The total energy of the harmonic oscillator

$$E = K + U = \frac{1}{2}m v_x^2 + \frac{1}{2}kx^2$$

$$= \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t+\phi) + \frac{1}{2}kA^2\cos^2(\omega t+\phi)$$

$$= \frac{1}{2}kA^2[\sin^2(\omega t+\phi)+\cos^2(\omega t+\phi)] \qquad \text{since } \omega^2 = \frac{k}{m}$$

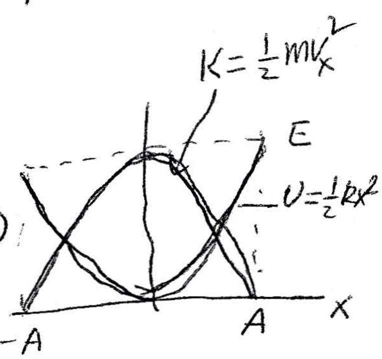
$$= \frac{1}{2}kA^2 = \text{constant.} \qquad (1)$$

The total energy becomes kinetic and the velocity becomes maximum when the particle passes through the eqlm point $(x=0)$

$$K_{max} = \frac{1}{2}m v_{max}^2 = \frac{1}{2}kA^2 = E \qquad \text{when } x=0$$

$$U_{max} = \frac{1}{2}k\cancel{x}_{max}^2 = \frac{1}{2}kA^2 = E \qquad \text{when } v=0$$

$\Rightarrow$ The total energy is potential when $v=0$, i.e when the displacement is maximum.

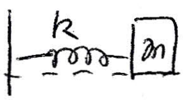


From Eqn (1) we can find v_x at any ~~time~~ x.

$$\frac{1}{2}m v_x^2 = \frac{1}{2}kA^2 - \frac{1}{2}kx^2 = \frac{1}{2}k(A^2-x^2)$$

$$\boxed{v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2-x^2}}$$

Example: Given total energy $E = 5\,J$, $k = 10\,\frac{N}{m}$, $m = 4\,kg$

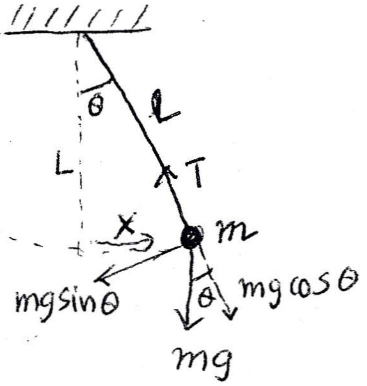


a) Find A: $E = \frac{1}{2}kA^2 \Rightarrow A = \sqrt{\frac{2E}{k}} = \sqrt{\frac{2(5)}{10}} = 1\,m$

b) v_{max}: $E = \frac{1}{2}m v_{max}^2 \Rightarrow v_{max} = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2(5)}{4}} = 1.58\,m/s$

c) Time period: $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{4}{10}} = 3.97\,s$

The Simple Pendulum



A simple pendulum consists of a point mass suspended by an inextensible weightless string. When pulled to one side and released, the pendulum executes SHM about the equilibrium position.

The restoring force

$$F_\theta = -mg\sin\theta \simeq -mg\theta = -mg\frac{x}{L}$$

$$ma_x = F_\theta \Rightarrow m\frac{d^2x}{dt^2} = -\frac{mg}{L}x \Rightarrow \frac{d^2x}{dt^2} + \frac{g}{L}x = 0$$

or $\frac{d^2x}{dt^2} + \omega^2 x = 0$ where $\omega = \sqrt{\frac{g}{L}}$

Solution: $x(t) = A\cos(\omega t + \delta)$

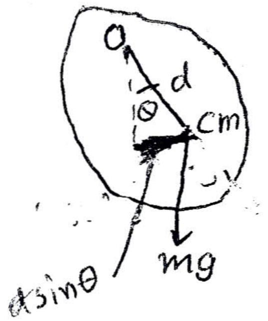
The period: $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$ independent of mass only depends on length L and g

$g = \left(\frac{2\pi}{T}\right)^2 L$. By measuring the time period and knowing the length of the pendulum g can be measured.

Example: Man enters a tall tower and notices a long pendulum extending from the ceiling to the floor. He can find the ~~length~~ height of the tower by measuring the period. Suppose he finds $T = 12.0\,s$, $L = ?$

$$L = \frac{gT^2}{4\pi^2} = \frac{(9.8\,m/s^2)(12.0\,s)^2}{4\pi^2} = 35.7\,m$$

The Physical Pendulum



Any solid body of any shape pivoted about a horizontal axis through O constitutes a physical pendulum. When the body is given a displacement from equilibrium, it undergoes SHM about equilibrium position. The weight gives it a torque about the axis of oscillation

$$\tau = -mgd\sin\theta \simeq -mgd\,\theta$$

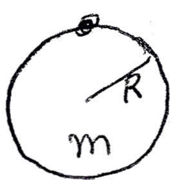
But $\quad I\alpha = \tau \quad$ where α is the angular acceleration $= \frac{d^2\theta}{dt^2}$ and I is the moment of inertia about O.

$$\Rightarrow \quad I\frac{d^2\theta}{dt^2} = -mgd\,\theta \quad \Rightarrow \quad \frac{d^2\theta}{dt^2} + \frac{mgd}{I}\theta = 0$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \quad \text{where } \omega = \sqrt{\frac{mgd}{I}}$$

The period: $\quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}$

Example: A circular sign of mass M and radius R is hung on a nail from a small loop located at one edge. Find the time period.



$\boxed{\text{See text for answer}}$

Example: Rigid rod $l = 0.6\,m$, $m = 2\,kg$. Pivoted at on end and undergoes oscillation

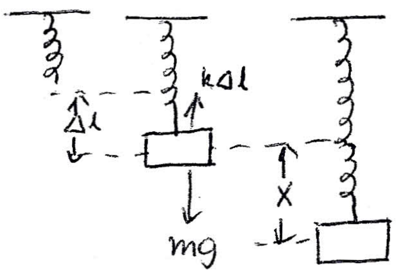
$$I_0 = I_{CM} + mh^2 = \frac{1}{12}ml^2 + m\left(\frac{l}{2}\right)^2 = \frac{1}{3}ml^2$$

$$= 0.24\ kg\,m^2$$

$$\omega = \sqrt{\frac{mgd}{I}} = \sqrt{\frac{3g}{2l}} = 4.95\ rad/s$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{4.95} = 1.27\ s$$

Vertical Simple Harmonic Motion.



Motion of a suspended body from a vertical spring:

At eqlm $\Sigma F_x = 0$

$$mg - k\Delta l = 0 \Rightarrow \boxed{k = \frac{mg}{\Delta l}}$$

If the mass is given further displacement x from eqlm, the restoring force $F = mg - k(\Delta l + x) = mg - k\Delta l - kx = -kx$

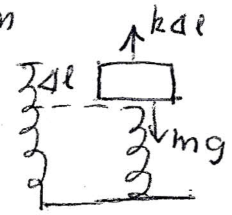
The mass will oscillate and the equation of motion is

$$m\frac{d^2x}{dt^2} = -kx \Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{where } \omega = \sqrt{\frac{k}{m}}$$

The same equation of motion as the horizontal spring. In this case the spring constant can be found from eqlm. condition

Example: Vertical SHM in an old car

An old car of mass $m = 1000\,kg$ has shock absorber worn out. When a 980 N person climbs slowly into the car, the car sinks by 2.8 cm. The moving car with person in it oscillates up and down in SHM. Model car ~~by~~ & person by a mass on a single spring and find frequency of oscillation



When man climbs slowly

$$k = \frac{mg}{\Delta l} = \frac{980\,N}{0.028\,m} = 3.5\times10^4\,\frac{N}{m}$$

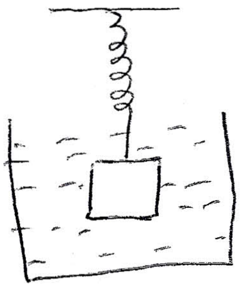
Total mass $M = m_{car} + m_{person} = 1000 + \frac{980}{9.8} = 1100\,kg$

$$T = 2\pi\sqrt{\frac{M}{k}} = 2\pi\sqrt{\frac{1100\,kg}{3.5\times10^4\,N/m}} = 1.11\,s$$

$$f = \frac{1}{T} = \frac{1}{1.115} = 0.90\,Hz$$

Damped Oscillation

So far we have considered an idealized situation where there is no damping force acting on an oscillator. But in almost all situations there is a damping force due to friction. The damping force is usually proportional to velocity. So the restoring force



on a mass-spring system

$$F_x = -kx - b\frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

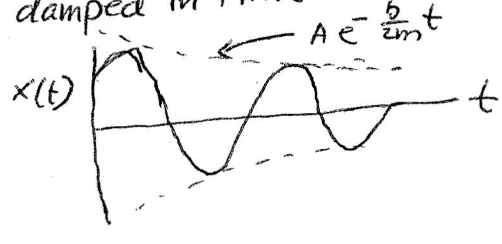
$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

When the resistive force is small i.e. $\frac{b^2}{4m} < k$, the soln is

$$x(t) = \left[A e^{-\frac{b}{2m}t}\right] \cos(\omega' t + \phi)$$

where $\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2} \simeq \omega$

Notice that the motion is still harmonic with slightly different angular frequency but the amplitude gets damped in time



Notice that the frequency of oscillation becomes zero when b becomes so large that

$$\frac{k}{m} - \frac{b^2}{4m^2} = 0 \quad \text{or} \quad b = 2\sqrt{mk}$$

When this condition is satisfied, the condition is called critical damping. When $b > 2\sqrt{mk}$ we have overdamping.

Example: $m = 1.5\,kg$, $b = 0.23\,kg/s$

a) At what time does the amplitude become 1/3 of original amplitude

$$A e^{-b/2m\,t} = \frac{1}{3}A \Rightarrow -\frac{b}{2m}t = -\ln 3 \Rightarrow t = \frac{2m\ln 3}{b}$$

$$= \frac{2\,(1.5\,kg)\ln 3}{0.23\,kg/s} = 14.33\,s$$

b) $T = \frac{2\pi}{\omega'} \simeq \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

$= 2.72\,s$