Undergraduate Lab for Fourier Analysis and Wavelet Analysis

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2010 AAPT/APS Winter Meeting
February 16, 2010 Tuesday
7:00 PM – 7:12 PM
GG01
Washington 1

Topics:

1. The Prelab
   a) Fourier Series
   b) Simple Fourier Analysis
2. The Lab
   a) Objectives
   b) Data Collection
3. Observations and Conclusions
   a) Fourier Analysis
   b) FFT
   c) Wavelet Analysis
4. Other Applications
   a) Mass on a Spring
   b) Temperature Data

Topics:

1. Prelab
   a) Fourier Series
   b) Simple Fourier Analysis

Fourier Series

Jean Baptiste Joseph Fourier
Born: 21 March 1768 in Auxerre, Bourgogne, France
Died: 16 May 1830 in Paris, France

According to Fourier, ANY periodic function can be represented as a sum of sine and cosine terms.

Fourier Analysis
A Real Simple Approximation

Assume an integer number of period which are equal to n*dt.

$$f(t)=\frac{1}{2}a_o+\sum_{n=1}^{\infty}\left[A_n\sin\left(2\pi\nu_n t\right)+B_n\cos\left(2\pi\nu_n t\right)\right]$$

$$A_n=\frac{2}{n*dt}\int_0^{n*dt} f(t)\sin\left(2\pi\nu_n t\right)dt$$

$$B_n=\frac{2}{n*dt}\int_0^{n*dt} f(t)\cos\left(2\pi\nu_n t\right)dt$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax)$$

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax)$$

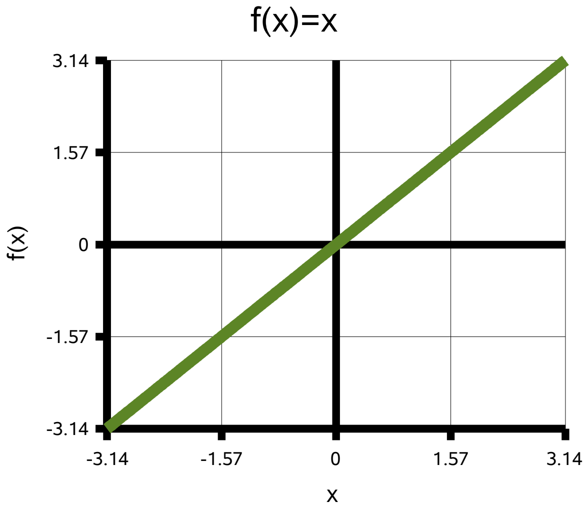
$$\int u\,dv = uv - \int v\,du$$ Integration by Parts.

$$f(x)=x \qquad -\pi \le x < \pi$$

$$f_o(x)=\sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{a}\right) \qquad -a \le x \le +a$$

$$f(x)=\sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{a}\right) \qquad 0 \le x \le a$$

$$b_k=\frac{2}{a}\int_0^a f(x)\sin\left(\frac{k\pi x}{a}\right)dx$$



Odd Function

$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx$$

$$u=f(x)=x \qquad du=dx$$

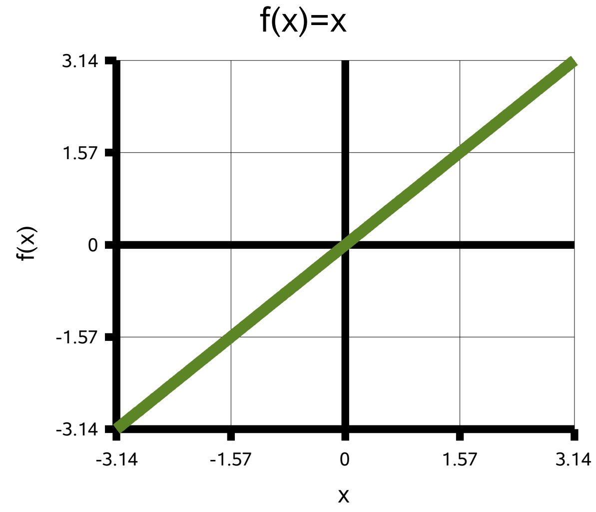
$$v=g(x)=\sin(kx) \qquad dv=k\cos(kx)\,dx$$

$$\int x\sin(kx)\,dx=\int kx\cos(kx)\,dx=x\sin(kx)+\int \sin(kx)\,dx$$

Example: Ramp

$$f(x)=x \quad -\pi \le x < \pi$$

Odd Function



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$$f_o(x)=\sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{a}\right) \qquad -a \le x \le +a$$

$$f(x)=\sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{a}\right) \qquad 0 \le x \le a$$

$$b_k=\frac{2}{a}\int_0^a f(x)\sin\left(\frac{k\pi x}{a}\right)dx$$

$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx$$

$$u=f(x)=x \qquad du=dx$$

$$dv=\sin(kx) \qquad v=\int dv=\int \sin(kx)\,dx=\frac{-1}{k}\cos(kx)$$

$$\int u\,dv=uv-\int v\,du$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx=\frac{1}{\pi}\left[(-x)\frac{1}{k}\cos(kx)-\int\left(\frac{-1}{k}\right)\cos(kx)\,dx\right]$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx=\frac{1}{\pi}\left[\frac{-\pi}{k}\cos(kx)+\frac{1}{k^2}\sin(kx)\right]_{-\pi}^{\pi}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx=\frac{1}{\pi}\left[\left(\frac{-\pi}{k}\cos(k\pi)+\frac{1}{k^2}\sin(k\pi)\right)-\left(\frac{-\pi}{k}\cos(k(-\pi))+\frac{1}{k^2}\sin(k(-\pi))\right)\right]$$

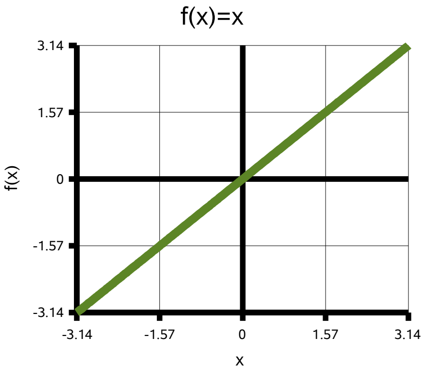
$$\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx=\frac{1}{\pi}\left[\left(\frac{-\pi}{k}\cos(k\pi)\right)-\left(\frac{-(-\pi)}{k}\cos(k(-\pi))\right)\right]=\left(\frac{-2}{k}\right)\cos(k\pi)$$

Example: Ramp

$$f_o(x)=\sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{a}\right) \qquad -a \le x \le +a$$

$$f(x)=\sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{a}\right) \qquad 0 \le x \le a$$

$$b_k=\frac{2}{a}\int_0^a f(x)\sin\left(\frac{k\pi x}{a}\right)dx$$



$$f(x)=x \qquad -\pi \le x < \pi$$

Odd Function

$$b_k=\left(\frac{-2}{k}\right)\cos(k\pi)$$

$$b_1=\left(\frac{-2}{1}\right)\cos(\pi)=\frac{2}{1}$$

$$b_2=\left(\frac{-2}{2}\right)\cos(2\pi)=\frac{-2}{2}$$

$$b_3=\left(\frac{-2}{3}\right)\cos(3\pi)=\frac{2}{3}$$

$$b_4=\left(\frac{-2}{4}\right)\cos(4\pi)=\frac{-2}{4}$$

$$b_5=\left(\frac{-2}{5}\right)\cos(5\pi)=\frac{2}{5}$$

$$F(x)=\sum_{k=1}^{\infty}\frac{2(-1)^{k+1}}{k}\sin(kx)$$

$$b_k=\left(\frac{-2}{k}\right)(-1)^{k+1}$$

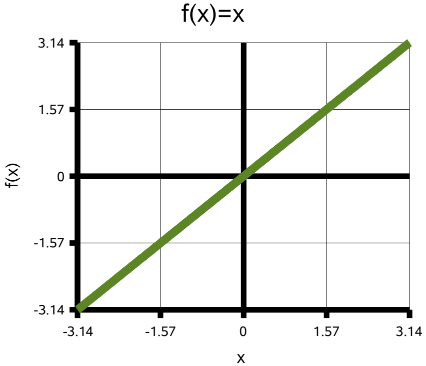
Example: Ramp

$$f_o(x)=\sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{a}\right) \qquad -a \le x \le +a$$

$$f(x)=\sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{a}\right) \qquad 0 \le x \le a$$

$$b_k=\frac{2}{a}\int_0^a f(x)\sin\left(\frac{k\pi x}{a}\right)dx$$

$$F(x)=\sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k}\sin(kx)$$



$$f(x)=x \qquad -\pi \le x < \pi$$

$$b_k=\left(\frac{-2}{k}\right)\cos(k\pi)$$

$$b_1=-\left(\frac{2}{1}\right)\cos(\pi)=+\frac{2}{1}$$

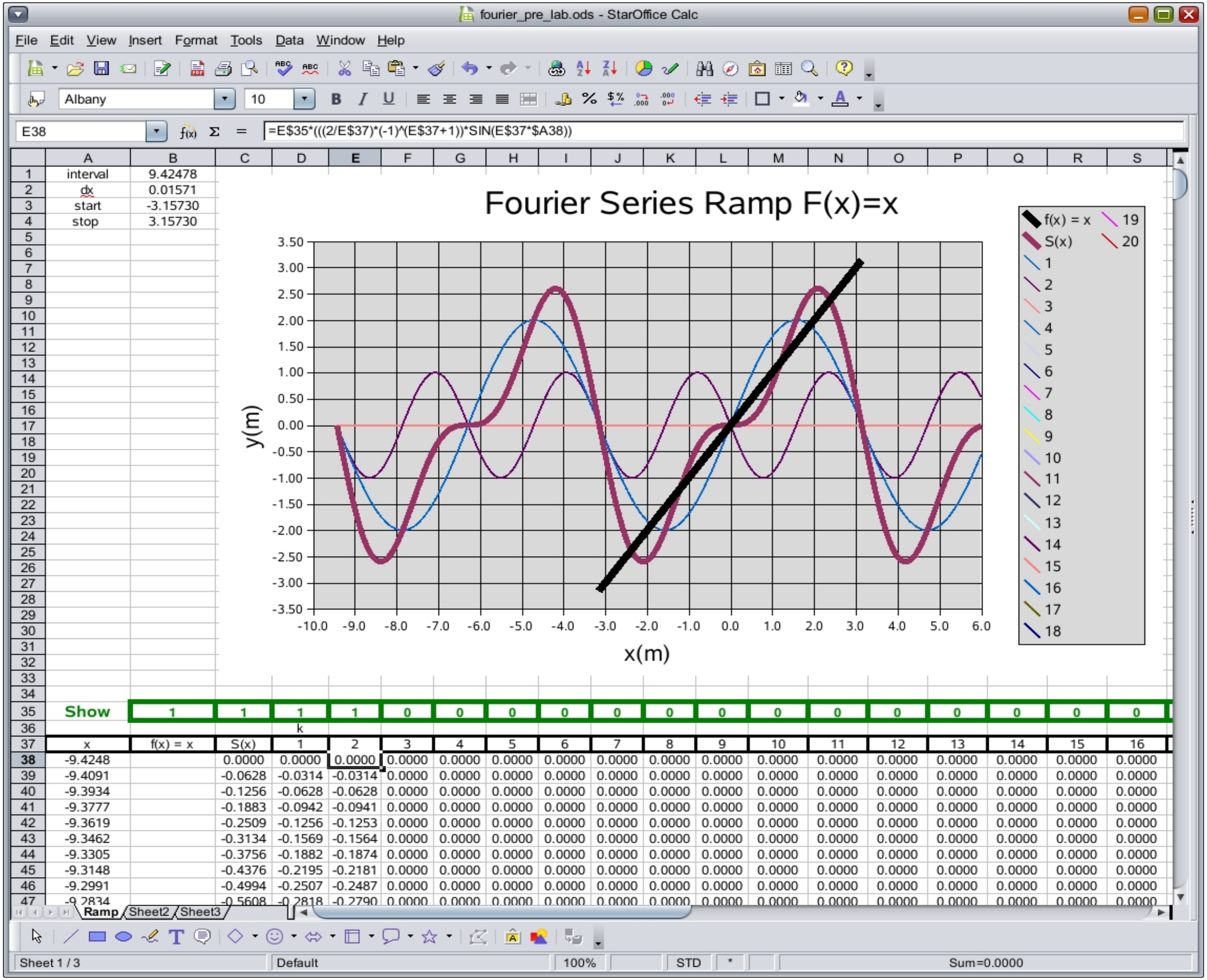
$$b_2=-\left(\frac{2}{2}\right)\cos(2\pi)=-\frac{2}{2}$$

$$b_3=-\left(\frac{2}{3}\right)\cos(3\pi)=+\frac{2}{3}$$

$$b_4=-\left(\frac{2}{4}\right)\cos(4\pi)=-\frac{2}{4}$$

$$b_5=-\left(\frac{2}{5}\right)\cos(5\pi)=+\frac{2}{5}$$

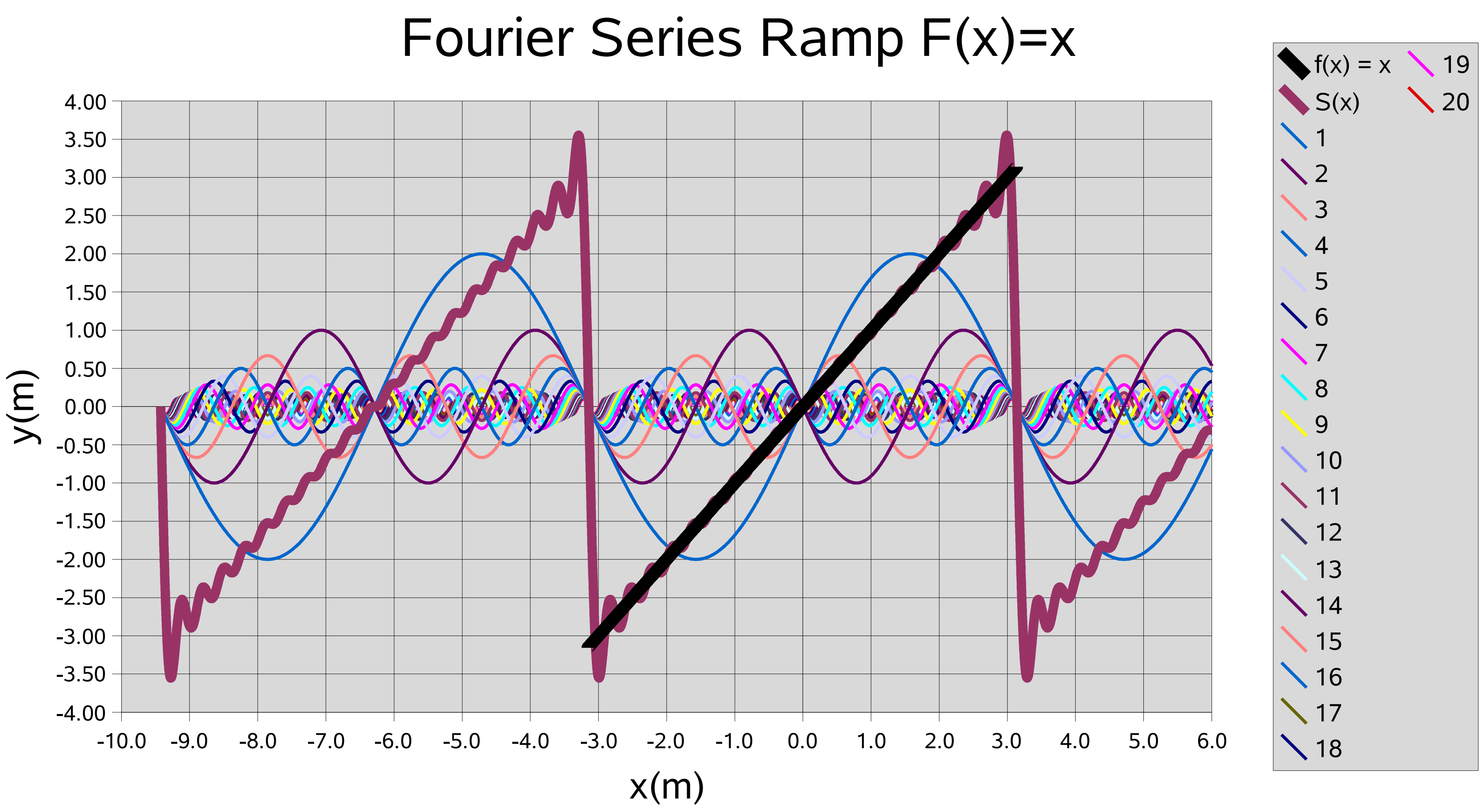
$$b_k=-\left(\frac{2}{k}\right)(-1)^{k+1}$$

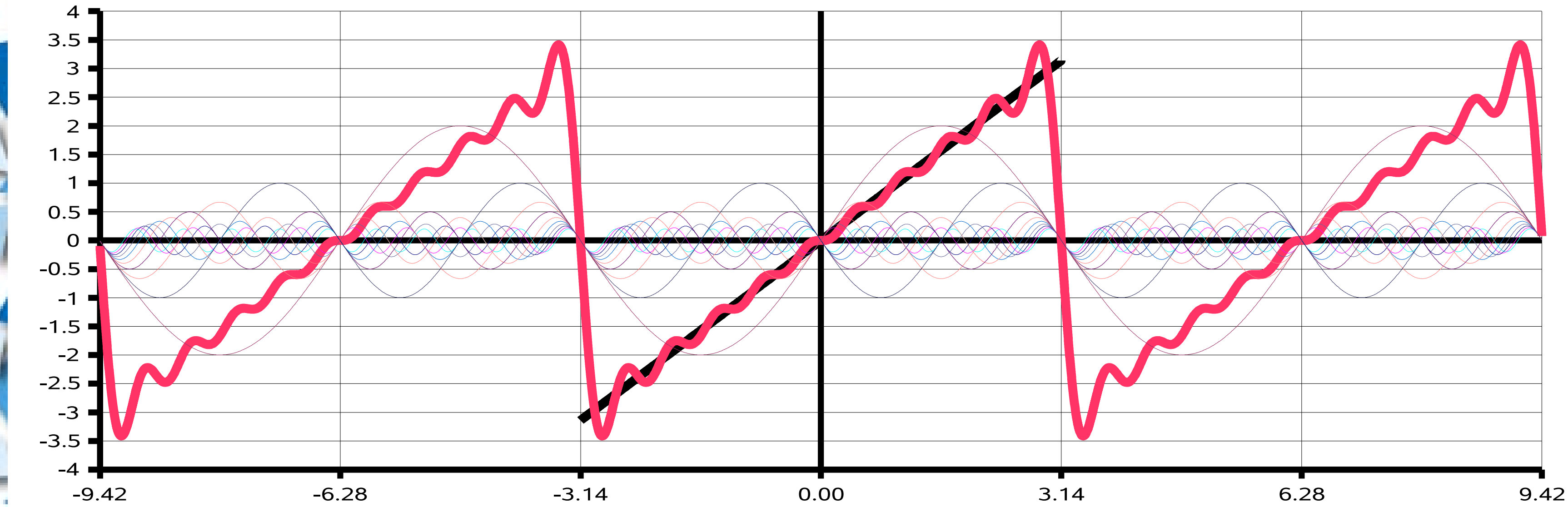


	A	B
1	interval	9.42478
2	dx	0.01571
3	start	-3.15730
4	stop	3.15730

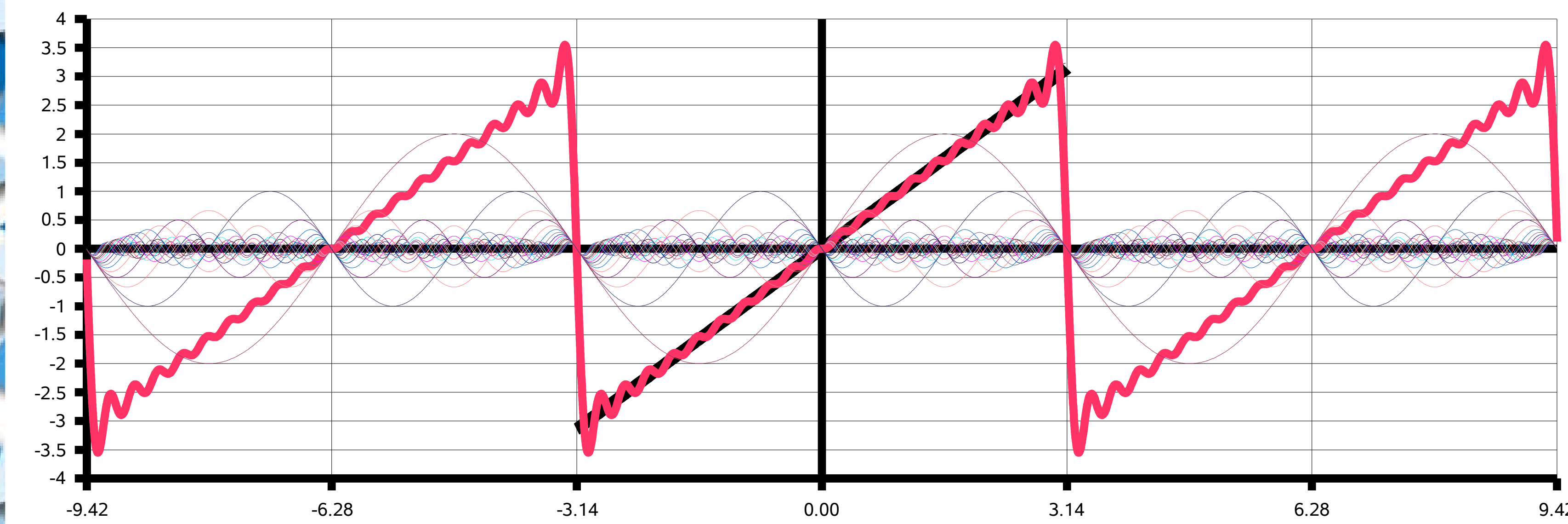
	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	Show	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36				k															
37	x	f(x) = x	S(x)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
38	-9.4248		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
39	-9.4091		-0.0628	-0.0314	-0.0314	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
40	-9.3934		-0.1256	-0.0628	-0.0628	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
41	-9.3777		-0.1883	-0.0942	-0.0941	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
42	-9.3619		-0.2509	-0.1256	-0.1253	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
43	-9.3462		-0.3134	-0.1569	-0.1564	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
44	-9.3305		-0.3756	-0.1882	-0.1874	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
45	-9.3148		-0.4376	-0.2195	-0.2181	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
46	-9.2991		-0.4994	-0.2507	-0.2487	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
47	-9.2834		-0.5608	-0.2818	-0.2790	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

$$S_{20}(x)=\sum_{k=1}^{20}\frac{2(-1)^{k+1}}{k}\sin(kx)$$





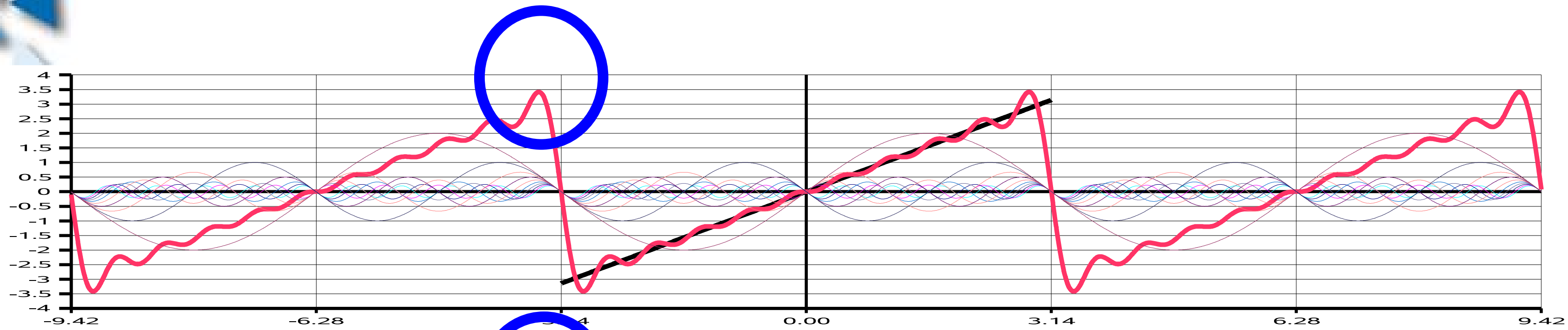
$S_{10}(x)$



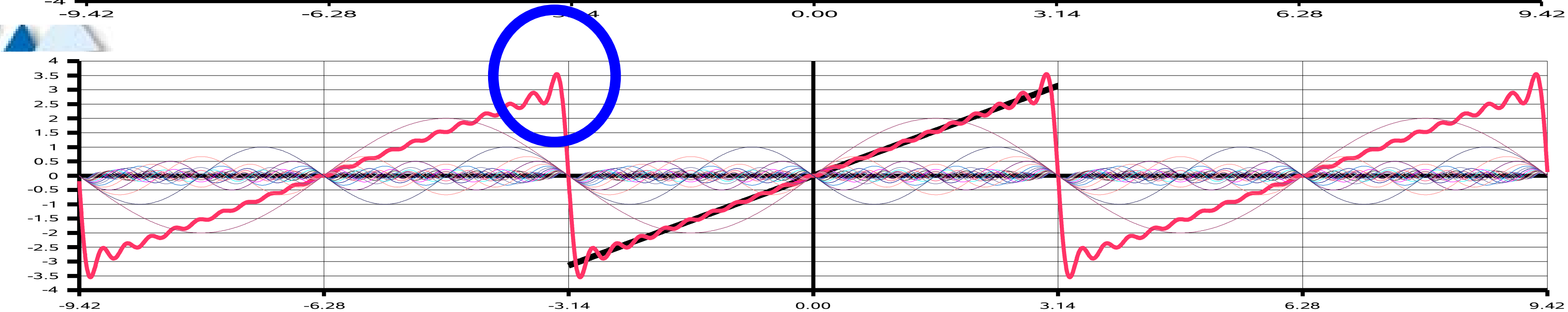
$S_{20}(x)$

The function $f(x) = x$ is not periodic. This is called the function's periodic extension: $\tilde{f}(x)$

Example: Ramp



$S_{10}(x)$



$S_{20}(x)$

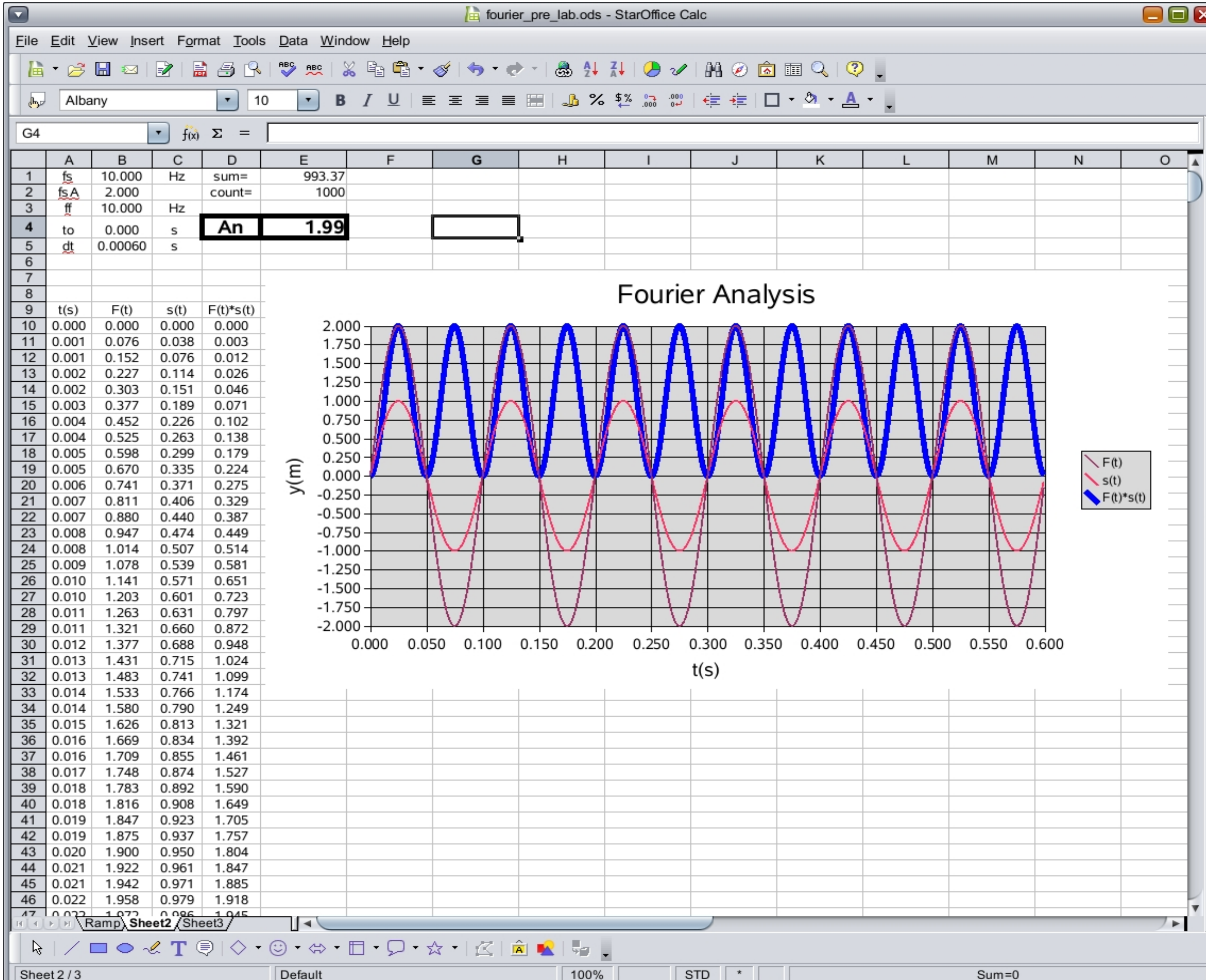
Gibb's Phenomenon:

•Notice that the accuracy of the approximation gets worse as S_N (x) as x approaches the discontinuity.

•Notice that the “blips” just before and just after the discontinuity in the graph of S_N (x), this is called the Gibb's Phenomenon.

•Notice that no matter how many terms are in the approximation, the amplitude of the “blips” remains approximately the same, but as more terms are added, the “wavelength” of the “blips” become shorter.

Simple Fourier Transform



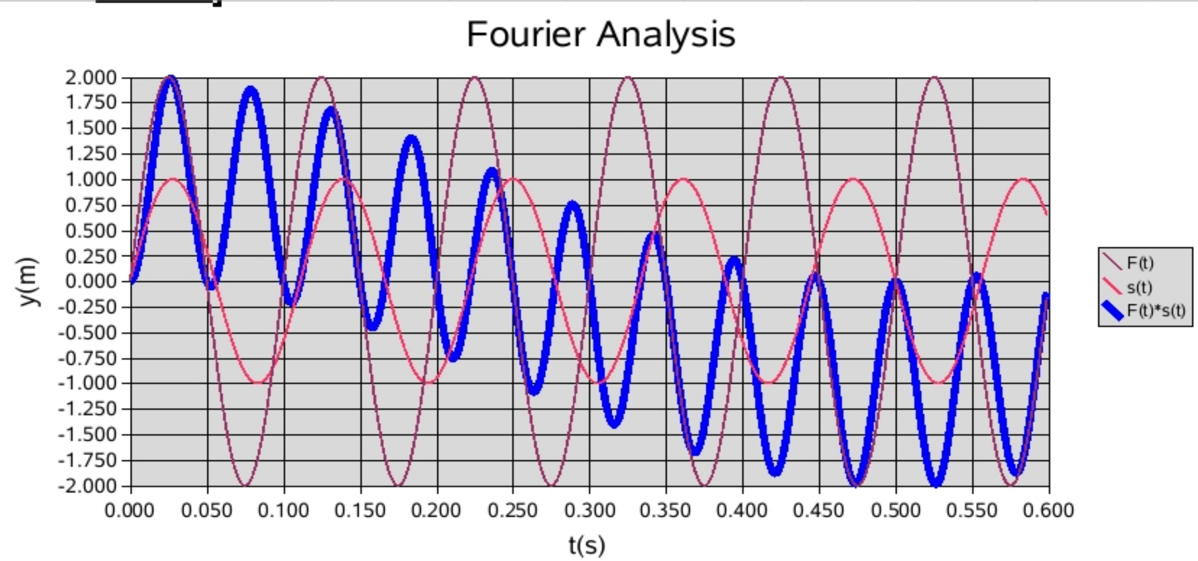
Simple Fourier Transform

fourier_pre_lab.ods - StarOffice Calc

	A	B	C	D	E
1	fs	10.000	Hz	sum=	-162.28
2	fsA	2.000		count=	1000
3	ff	9.000	Hz		
4	to	0.000	s	**An**	**0.32**
5	dt	0.00060	s		
6					
7					
8					
9	t(s)	F(t)	s(t)	F(t)*s(t)	
10	0.000	0.000	0.000	0.000	
11	0.001	0.076	0.034	0.003	
12	0.001	0.152	0.068	0.010	
13	0.002	0.227	0.102	0.023	
14	0.002	0.303	0.136	0.041	
15	0.003	0.377	0.170	0.064	
16	0.004	0.452	0.204	0.092	
17	0.004	0.525	0.237	0.124	
18	0.005	0.598	0.270	0.161	
19	0.005	0.670	0.303	0.203	
20	0.006	0.741	0.335	0.248	
21	0.007	0.811	0.367	0.298	
22	0.007	0.880	0.399	0.351	
23	0.008	0.947	0.430	0.407	
24	0.008	1.014	0.460	0.467	
25	0.009	1.078	0.490	0.529	
26	0.010	1.141	0.520	0.593	
27	0.010	1.203	0.549	0.660	
28	0.011	1.263	0.577	0.728	
29	0.011	1.321	0.604	0.798	
30	0.012	1.377	0.631	0.869	
31	0.013	1.431	0.657	0.941	
32	0.013	1.483	0.683	1.012	
33	0.014	1.533	0.707	1.084	
34	0.014	1.580	0.731	1.155	
35	0.015	1.626	0.754	1.226	
36	0.016	1.669	0.776	1.295	
37	0.016	1.709	0.797	1.362	
38	0.017	1.748	0.817	1.428	
39	0.018	1.783	0.836	1.492	
40	0.018	1.816	0.855	1.552	
41	0.019	1.847	0.872	1.610	
42	0.019	1.875	0.888	1.665	
43	0.020	1.900	0.903	1.716	
44	0.021	1.922	0.917	1.763	
45	0.021	1.942	0.930	1.807	
46	0.022	1.958	0.942	1.846	

Ramp | Sheet2 | Sheet3

Sheet 2 / 3 | Default | 100% | STD | Sum=0



Topics:

2. Objectives and Procedure

a) Objectives

b) Data Collection

Objectives:

- To investigate sound waves.

- To gain an understanding of Fourier by analyzing sound recordings made of tuning forks.

- To introduce Wavelet Analysis.

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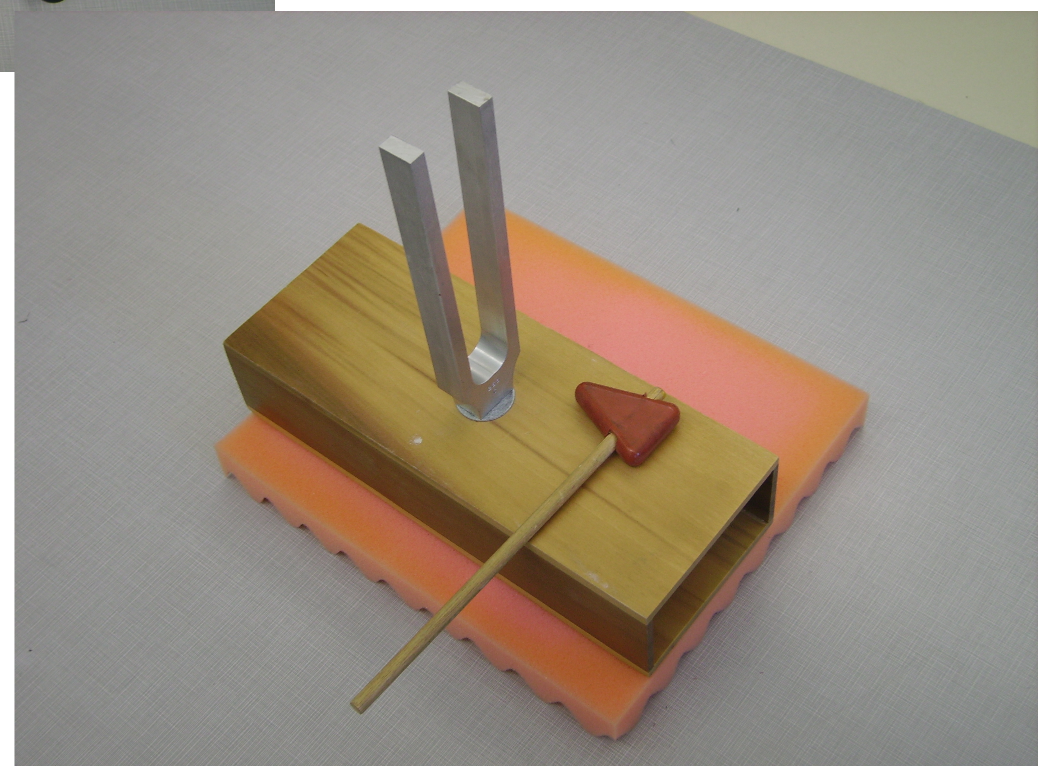
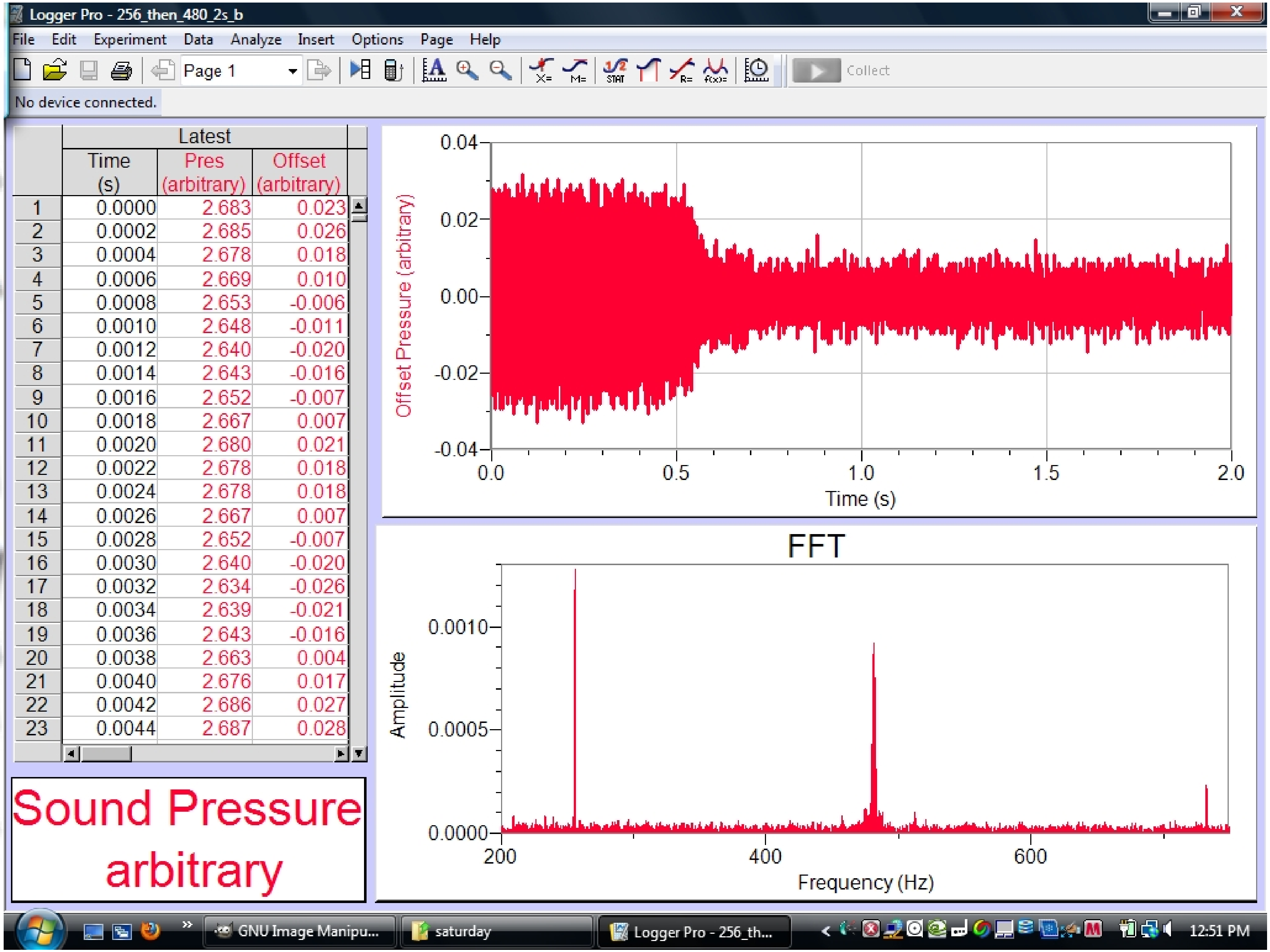




	Latest		
	Time (s)	Pres (arbitrary)	Offset (arbitrary)
1	0.0000	2.683	0.023
2	0.0002	2.685	0.026
3	0.0004	2.678	0.018
4	0.0006	2.669	0.010
5	0.0008	2.653	-0.006
6	0.0010	2.648	-0.011
7	0.0012	2.640	-0.020
8	0.0014	2.643	-0.016
9	0.0016	2.652	-0.007
10	0.0018	2.667	0.007
11	0.0020	2.680	0.021
12	0.0022	2.678	0.018
13	0.0024	2.678	0.018
14	0.0026	2.667	0.007
15	0.0028	2.652	-0.007
16	0.0030	2.640	-0.020
17	0.0032	2.634	-0.026
18	0.0034	2.639	-0.021
19	0.0036	2.643	-0.016
20	0.0038	2.663	0.004
21	0.0040	2.676	0.017
22	0.0042	2.686	0.027
23	0.0044	2.687	0.028

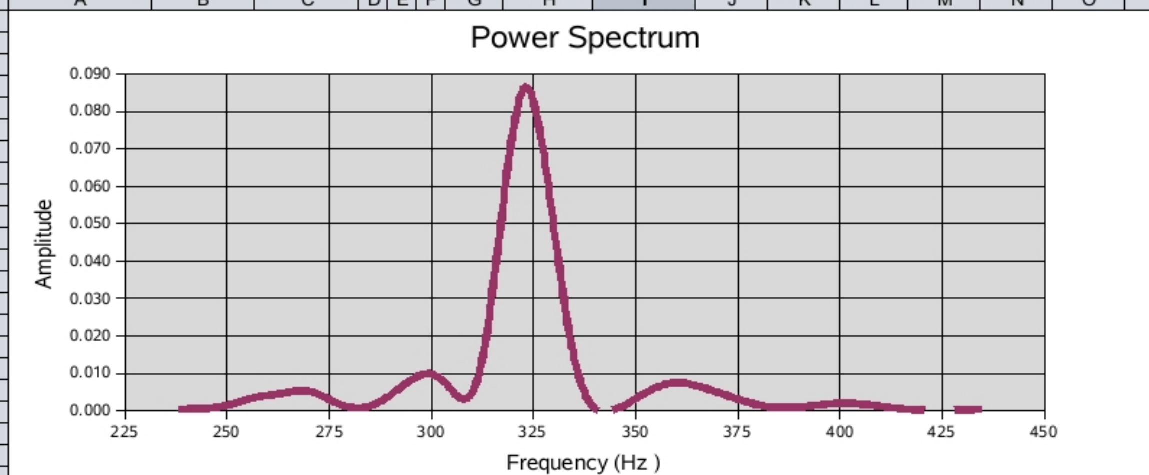
Topics:

3. Observations and Conclusions

a) Fourier Analysis
b) FFT
c) Wavelet Analysis

a)Fourier Analysis – Using a spreadsheet.

fourier_fft.ods - StarOffice Calc



	A	B	C	G	H	I	J	K	L	M	N	O	P	Q
24					f	10000.0	Hz							
25					f Nyquist	20000.0	Hz							
26					At	3.0100000								
27														
28	Vernier	Format	2.000		dt	0.0001000								
29	256_1sec.cmbl	40208.000	0.807		N	10001								
30	Latest			n	42	41	40	39	38	37	36	35	34	33
31	Time	Sound	Pressure		238.1	243.9	250	256.41	263.16	270.27	277.78	285.71	294.12	303.03
32	t	Pres	Offset											
33	s	arbitrary	arbitrary	3.01										
34														
35	0.000	2.915	0.259	2.91	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
36	0.000	2.955	0.299	2.95	1.326	0.138	0.141	0.145	0.148	0.152	0.156	0.161	0.166	0.171
37	0.000	2.987	0.331	2.99	2.650	0.301	0.308	0.316	0.324	0.332	0.341	0.350	0.360	0.370
38	0.000	3.005	0.349	3	3.924	0.467	0.478	0.489	0.501	0.513	0.526	0.540	0.554	0.569
39	0.000	3.010	0.354	3.01	5.103	0.614	0.627	0.641	0.655	0.670	0.686	0.702	0.719	0.736
40	0.001	2.999	0.343	3	6.139	0.717	0.731	0.745	0.760	0.776	0.792	0.808	0.825	0.842
41	0.001	2.974	0.319	2.97	7.000	0.763	0.777	0.790	0.804	0.817	0.831	0.845	0.859	0.873
42	0.001	2.932	0.276	2.93	7.642	0.730	0.741	0.751	0.761	0.771	0.781	0.791	0.800	0.808
43	0.001	2.878	0.222	2.88	8.064	0.630	0.637	0.643	0.649	0.655	0.660	0.664	0.667	0.669
44	0.001	2.816	0.160	2.82	8.263	0.473	0.476	0.479	0.481	0.482	0.482	0.482	0.480	0.477
45	0.001	2.748	0.093	2.75	8.250	0.280	0.280	0.280	0.279	0.278	0.276	0.273	0.269	0.265
46	0.001	2.674	0.019	2.67	8.026	0.056	0.055	0.055	0.054	0.053	0.053	0.051	0.050	0.048
47	0.001	2.603	-0.052	2.6	7.639	-0.152	-0.150	-0.147	-0.144	-0.140	-0.136	-0.131	-0.125	-0.119

Sheet1_2 / Sheet1 / Sheet2 / Sheet

Sheet 1 / 4 | Default | 100% | STD | * | Sum=0

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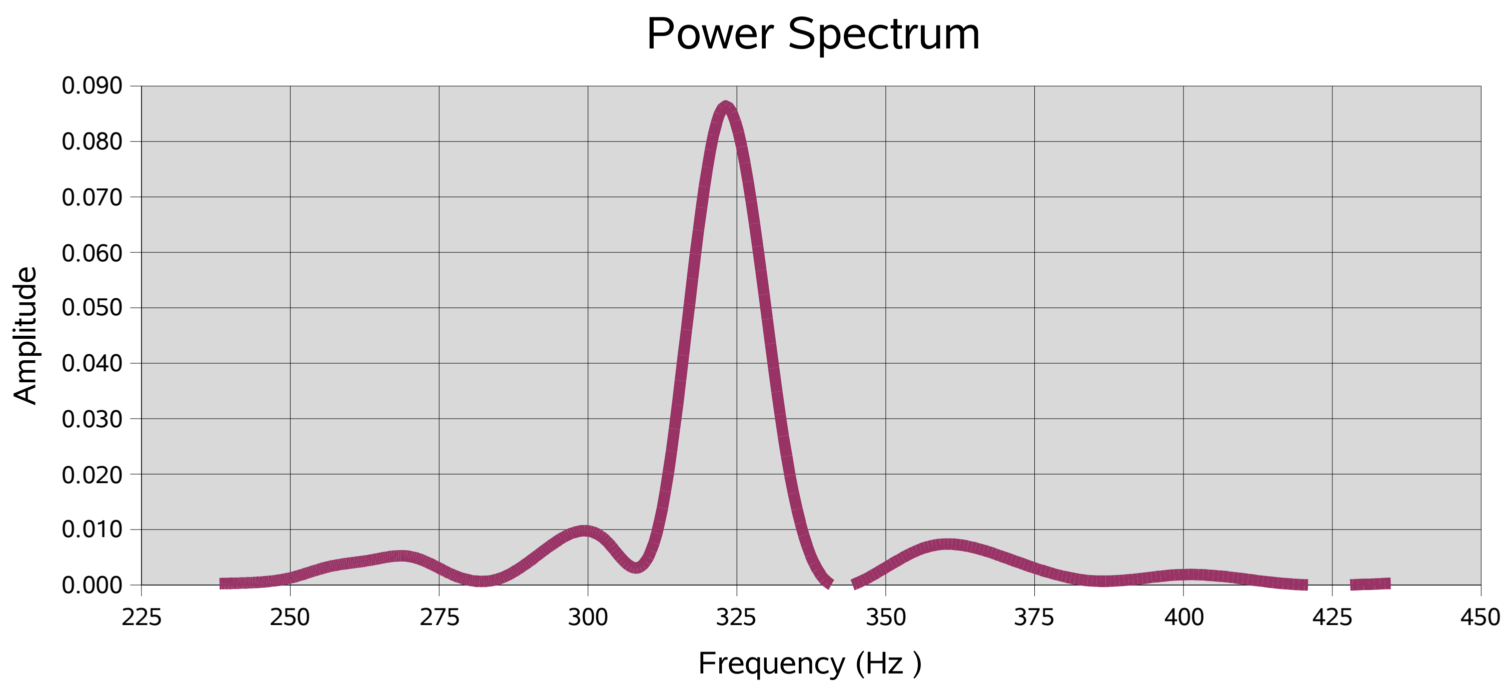
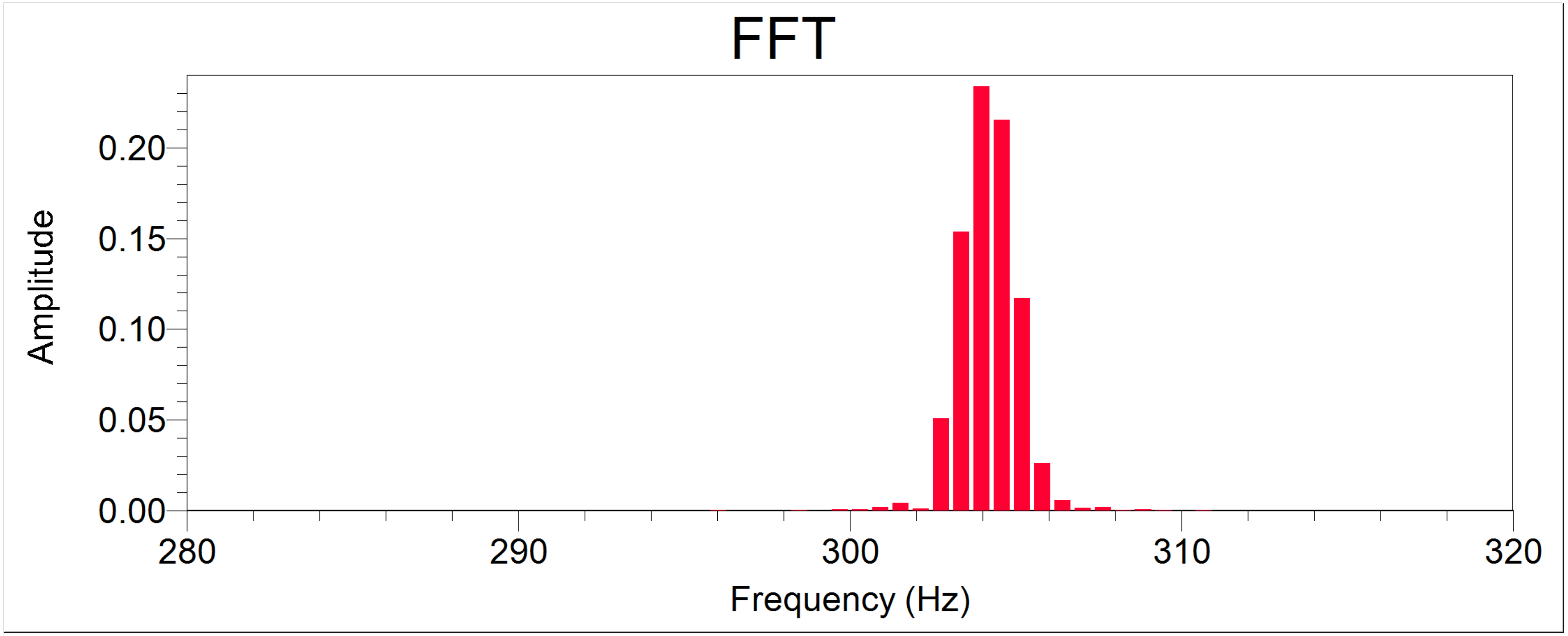
a)Fourier Analysis – Using a spreadsheet.



a)Fourier Analysis – Vernier FFT.



A Simple Code to do Fourier Analysis

```
/********** Do Cosine and Sine  Series *************/
  for(i=0; i<= n/2; i++ )
  {
    suma = 0.0;
    sumb = 0.0;
    for( j=0; j <= n; j++)
    {
      sumb += r[j]*cos(2.0*pi*f[i]*j*dt)*dt;
      suma += r[j]*sin(2.0*pi*f[i]*j*dt)*dt;
    }
    b[i]= (2.0/(n*dt))*sumb;
    a[i]= (2.0/(n*dt))*suma;
  }
/*****************************************************/
```

```
#include <math.h>
#include <stdlib.h>
#include <stdio.h>
#include <string.h>
/* cc -fast fourier_file.c -lm -v -o fourier_file */
void main ()
{
  int i, j;
  FILE *fpin, *fpout;
   float pi = 4.0*atan(1.0);
   float *a, *b, *f;
   float tdate, tx, ttemp, tdewpt;
   float *r;
   float azero, suma, sumb;
   char filein[40], fileout[40], ftype[5];
   int n=0;
 /***************************************************************************/
  strcpy( filein, "3087311062787_w.txt" );
  strcpy( fileout, "3087311062787_w_fourier.txt" );
    if (( fpin = fopen( filein, "r")) != NULL )
    {
       fprintf(stderr, "\n Successful opening of %s in mode r. \n", filein);
    } else {
       fprintf(stderr, "\n Error opening of %s in mode r. \n", filein);
    }
    while ( !feof(fpin) )
    {
       fscanf( fpin,"%f", &tx);
       n++;
    }
    fclose(fpin);
    printf(" Number of Points Available: %d \n", n);
    printf("Enter number of Points to use:");
    scanf("%d", &n);
    printf("\n");
    if (( fpin = fopen( filein, "r")) != NULL )
    {
       fprintf(stderr, "\n Successful opening of %s in mode r. \n", filein);
    } else {
       fprintf(stderr, "\n Error opening of %s in mode r. \n", filein);
    }
 /***************************************************************************/
```

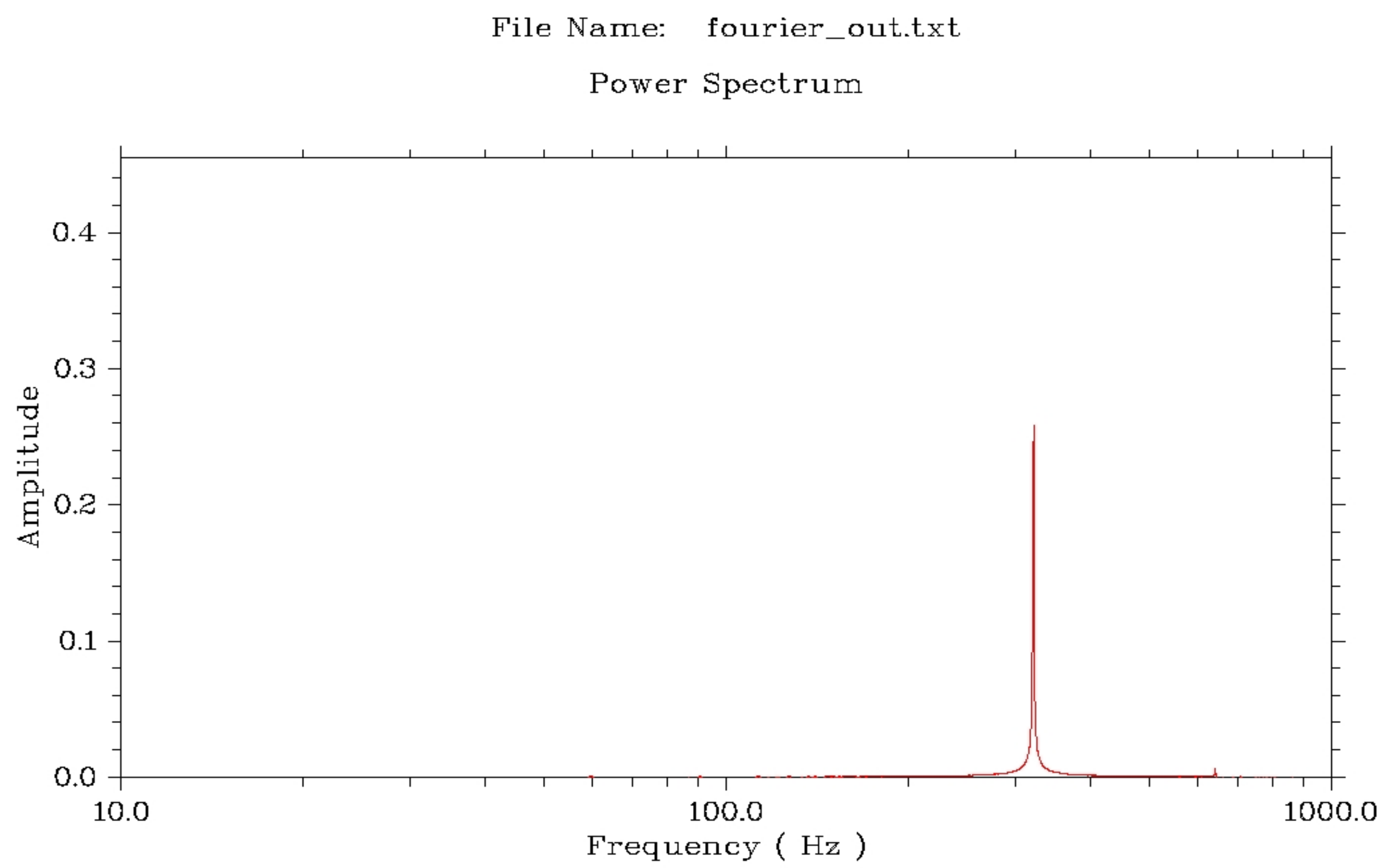
```
printf(" **** Begin setting up arrays. **** \n");
a=(float *)malloc(n*sizeof(float));
b=(float *)malloc(n*sizeof(float));
r=(float *)malloc(n*sizeof(float));
f=(float *)malloc(n*sizeof(float));
float x[n];
float dt;
dt = 1.0 / (365.25 * 24.0);
printf("dt = %f \n",  dt);
printf(" **** End setting up arrays. **** \n");
 if (( fpin = fopen( filein, "r")) != NULL )
 {
    fprintf(stderr, "\n Successful opening of %s in mode r. \n", filein);
 } else {
    fprintf(stderr, "\n Error opening of %s in mode r. \n", filein);
 }

 for(i = 0; i<= n-1 ; i++)
 {
    fscanf( fpin,"%f", &r[i]);
    printf("%d %f\n",i, r[i]);
 }
 fclose(fpin);
 printf(" **** End Importing Data     **** \n");
```

```
/*******************    Subtract Average ( DC Component ) ****************/
  azero = 0.0;
 for(i = 0; i <= n-1; i++ )
 {
    azero = azero + r[i];
 }
 azero = azero / n;
 printf("azero: %f \n", azero);
 for(i = 0; i <= n-1; i++ )
 {
    r[i] = r[i] - azero;
    f[i] = ((float)i+1.0)/((float)n* dt);
 }
printf(" **** End Remove DC     **** \n");
/*******************************************************************/
/******************* Do Cosine and Sine  Series **********************/
 for(i=0; i<= n/2; i++ )
 {
   suma = 0.0;
   sumb = 0.0;
   for( j=0; j <= n; j++)
   {
     sumb += r[j]*cos(2.0*pi*f[i]*j*dt)*dt;
     suma += r[j]*sin(2.0*pi*f[i]*j*dt)*dt;
   }
   b[i]= (2.0/(n*dt))*sumb;
   a[i]= (2.0/(n*dt))*suma;
 }
/*******************************************************************/
```

```
    printf("azero: %f \n", azero);
    if (( fpout = fopen( fileout, "w")) != NULL )
     {
       fprintf(stderr, "\n Successful opening of %s in mode w. \n", fileout);
      } else {
        fprintf(stderr, "\n Error opening of %s in mode w. \n", fileout);
      }
        for(i=0; i <= n/2 ; i++)
      {
          fprintf(fpout,"%f %f %f %f \n",f[i], r[i], a[i], b[i]);
      }

      fclose(fpout);
     free(f);
     free(a);
     free(b);
     free(r);
  }
```


File Name: fourier_out.txt
Power Spectrum
0.4
0.3
0.2
0.1
0.0
Amplitude
10.0
100.0
1000.0
Frequency (Hz)

Wavelets

Some Applications of Wavelets:

•Jean Morlet (1984), a geologist analyzing seismic signals, developed what are now known as 'Morlet wavelets”.

•Wavelets are central to the new JPEG-2000 digital image standard.

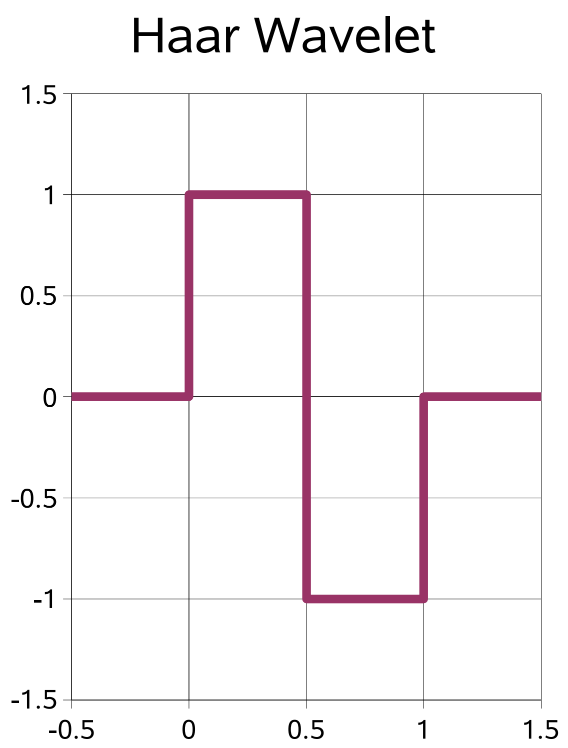
•WSQ method that the FBI uses to compress its fingerprint database.

•De-noising data.

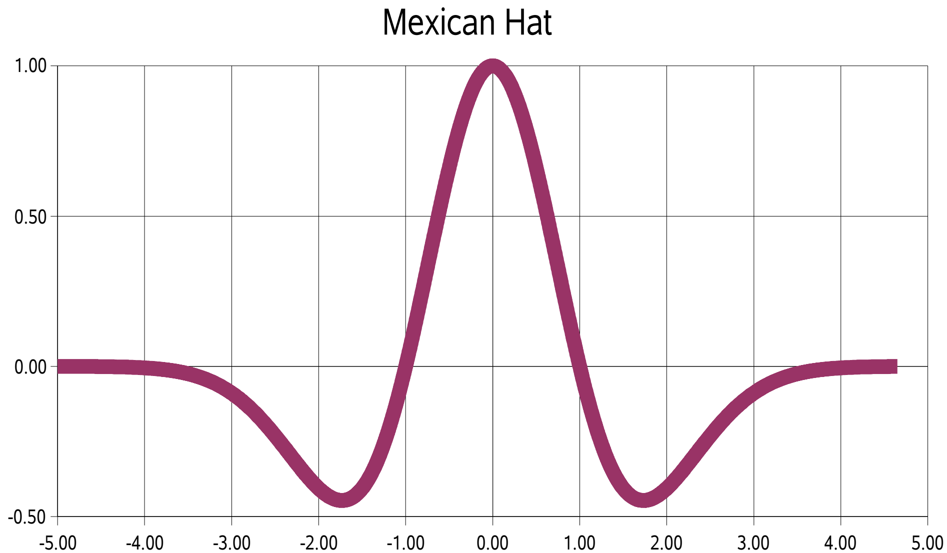
•Image Compression.

•Data Analysis

Sample of Wavelets:



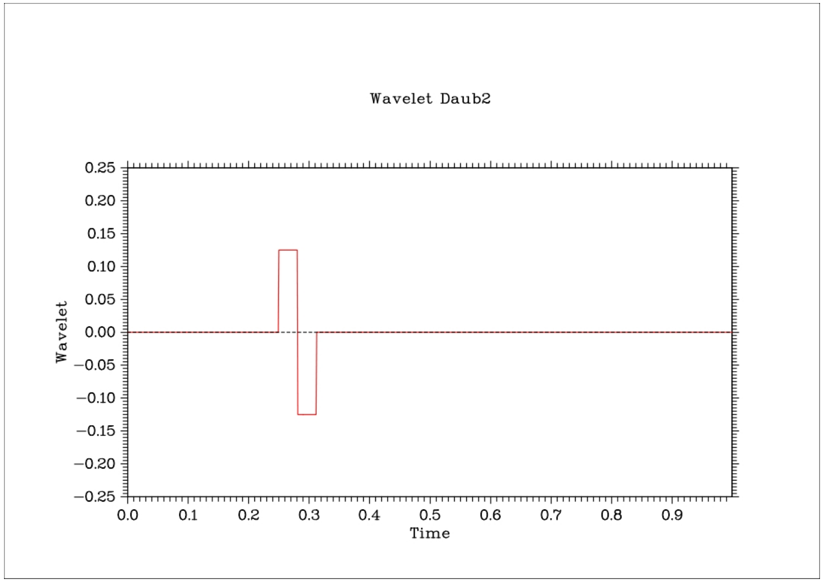
Haar Wavelet

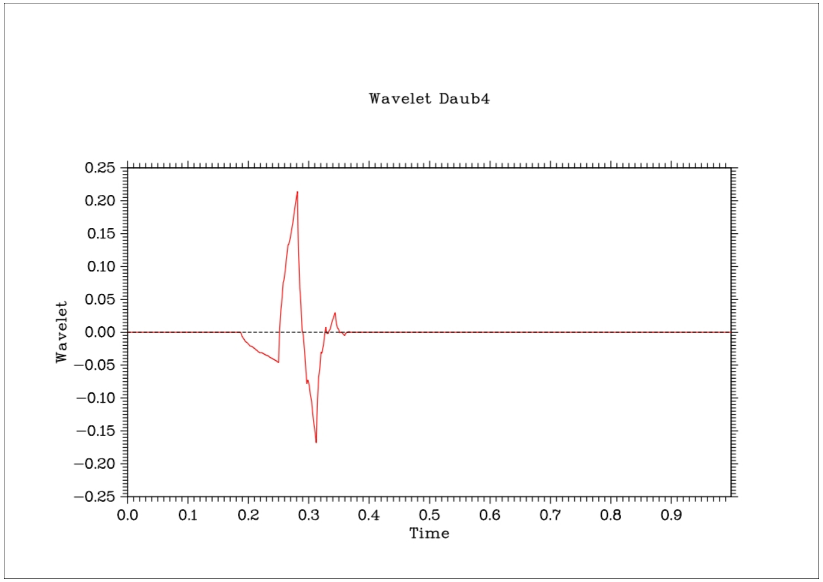


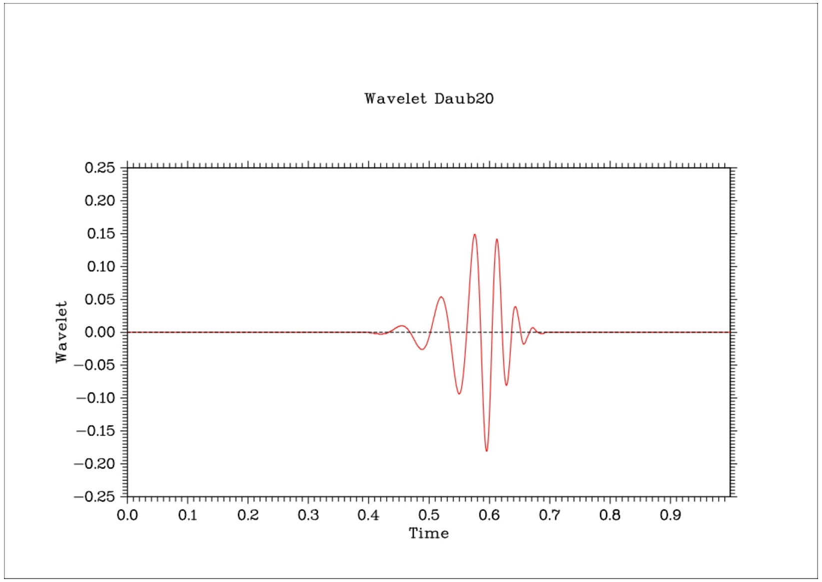
Mexican Hat Wavelet

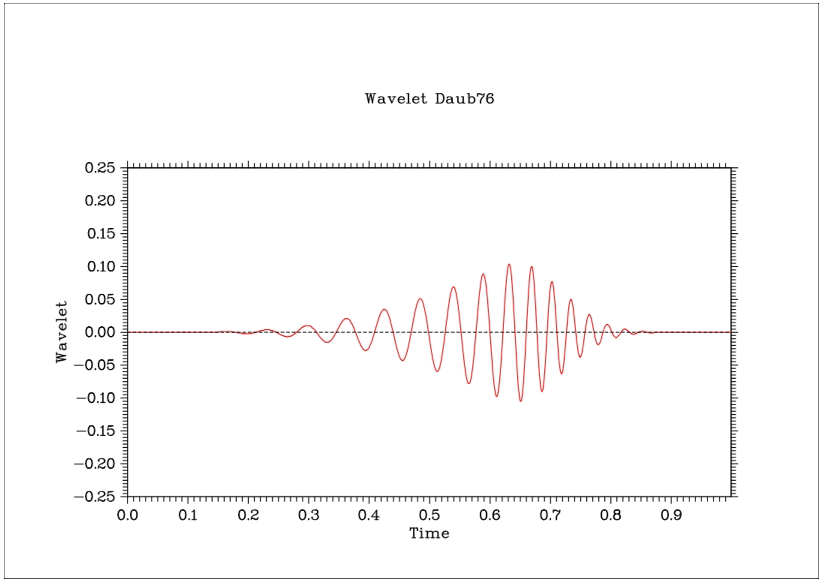
$$\Psi(t) = \left(\frac{1}{\sqrt{2\pi\sigma^3}}\right)\left(\left(1 - \frac{t^2}{\sigma^2}\right) e^{\left(\frac{-t^2}{2\sigma^2}\right)}\right)$$

Sample of Wavelets : Daubechies









Continuous Wavelet Transform

In General ….

Continuous Wavelet Transform

$$\gamma(\lambda,\tau)=\int f(t)\,\Psi^{*}_{\lambda,\tau}(t)\,dt$$

Wavelet Transform

$$f(t)=\int\int \gamma(\lambda,\tau)\,\Psi_{\lambda,\tau}(t)\,d\tau\,d\lambda$$

Inverse Wavelet Transform

$$\Psi_{\lambda,\tau}(t)=\frac{1}{\sqrt{\lambda}}\Psi\left(\frac{t-\tau}{\lambda}\right)$$

Wavelet – generated by scaling and translation of a *Mother Wavelet* , $\Psi(t)$

Discrete Wavelet Transform

In General ….

Discrete Wavelet Transform

$$f(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \gamma_{j,k} \Psi_{j,k}(t)$$

Function Expansion

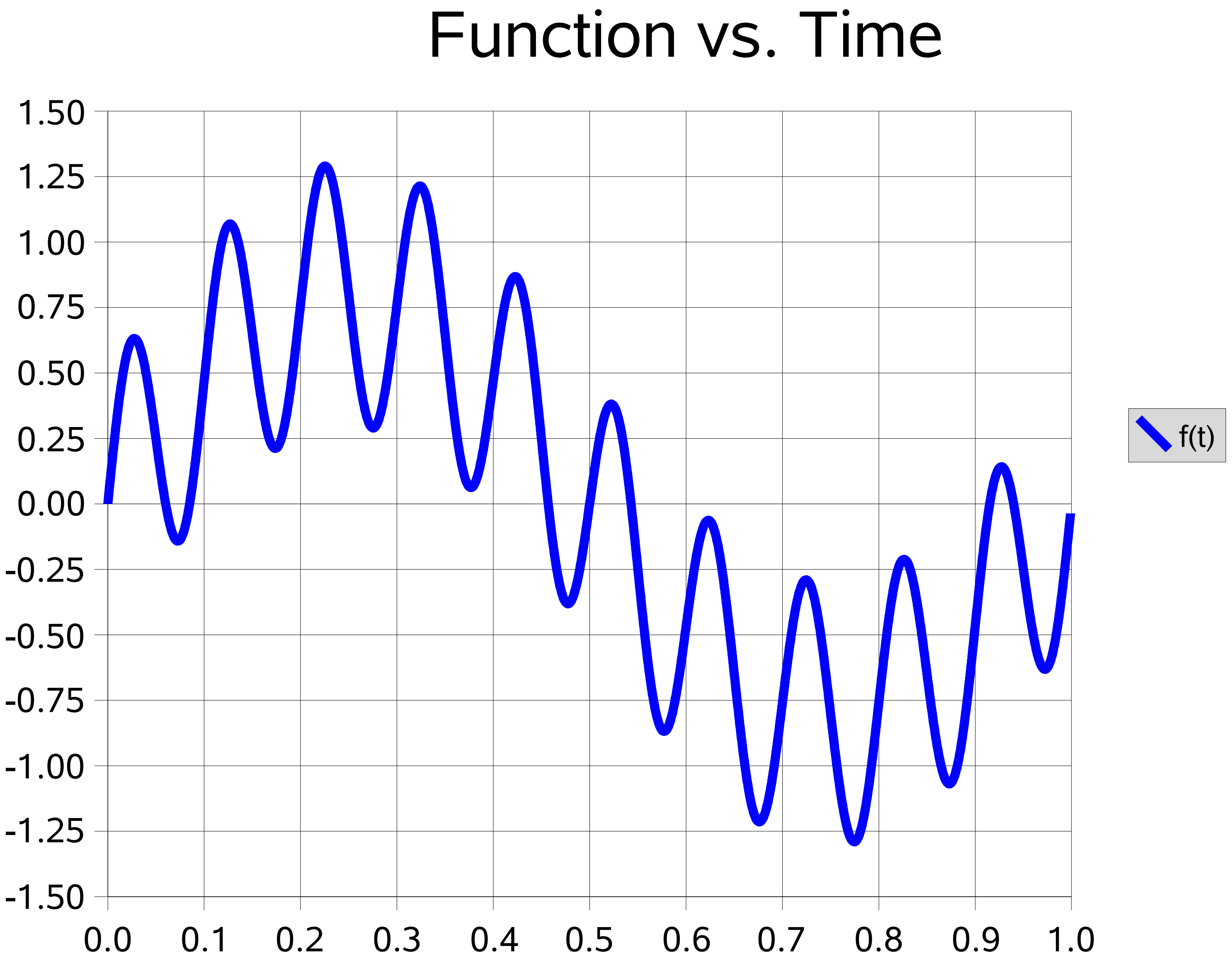
$$\Psi_{j,k}(t) = s^{j/2} \Psi\left(2^j t - k\right)$$

Basis Set

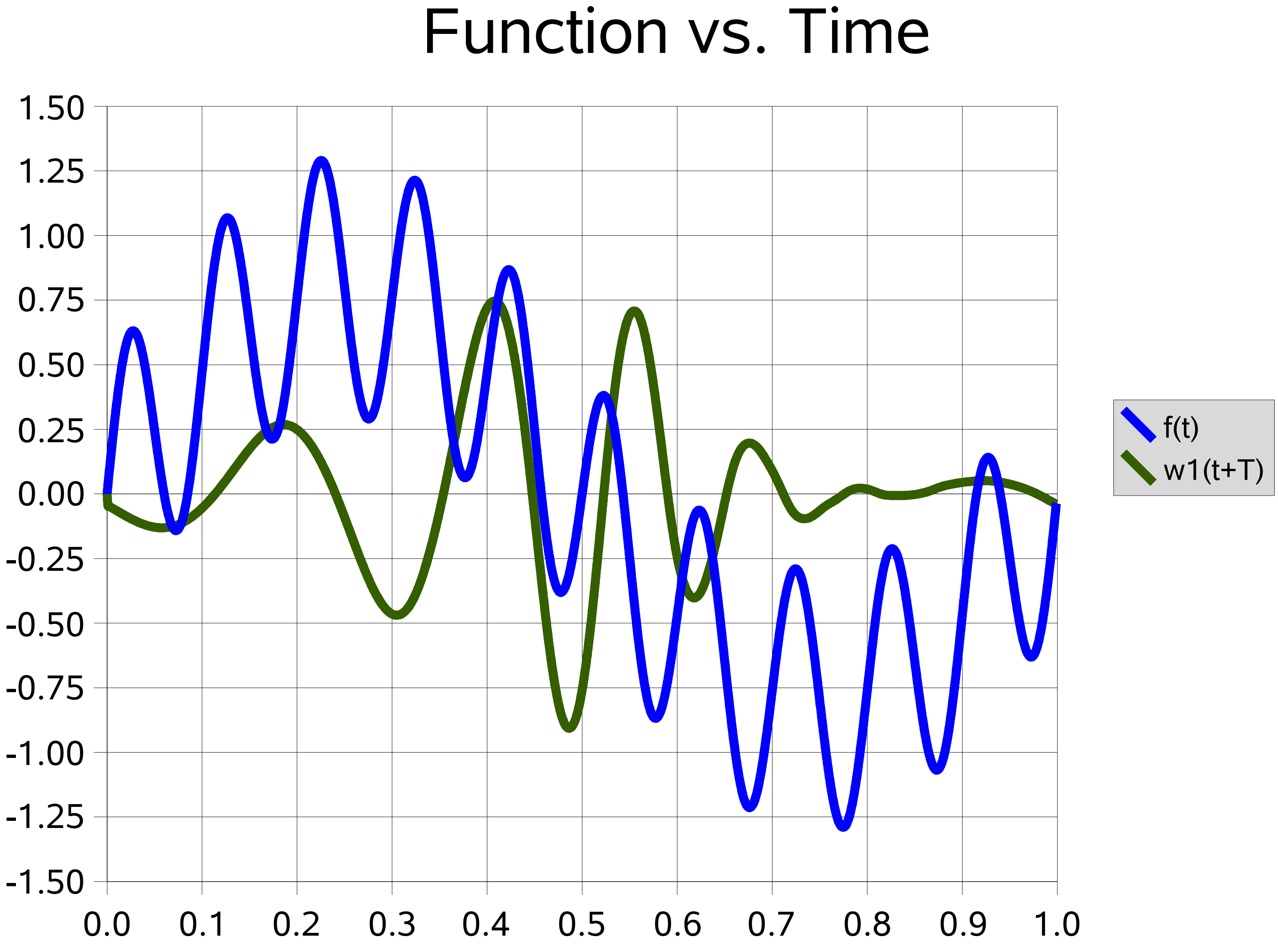
$$\gamma = \left\langle \Psi_{i,j} \| f \right\rangle = \int_{-\infty}^{\infty} f(t) \Psi_{j,k}(t) dt$$

Wavelet Transform

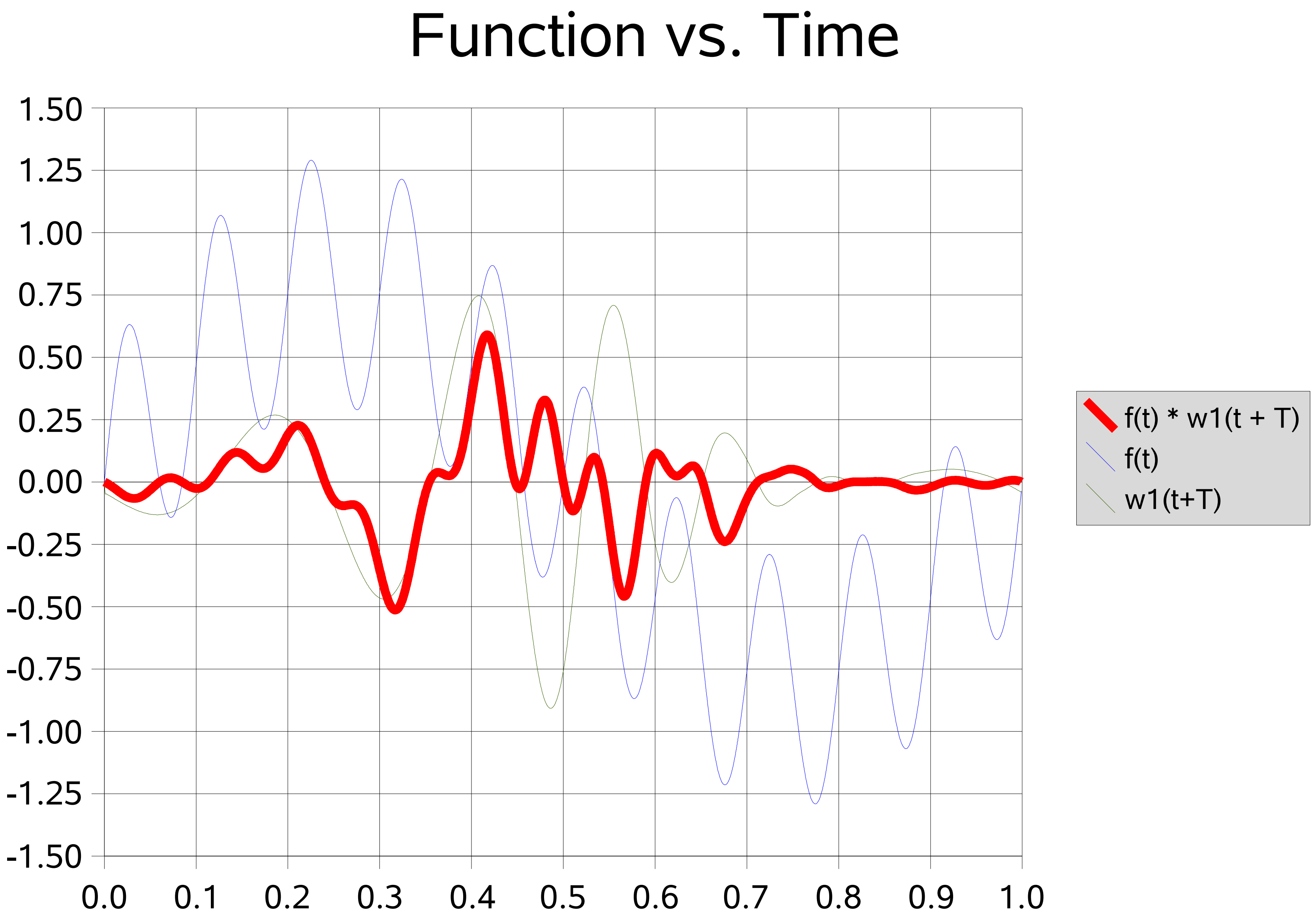
Wavelet Analysis



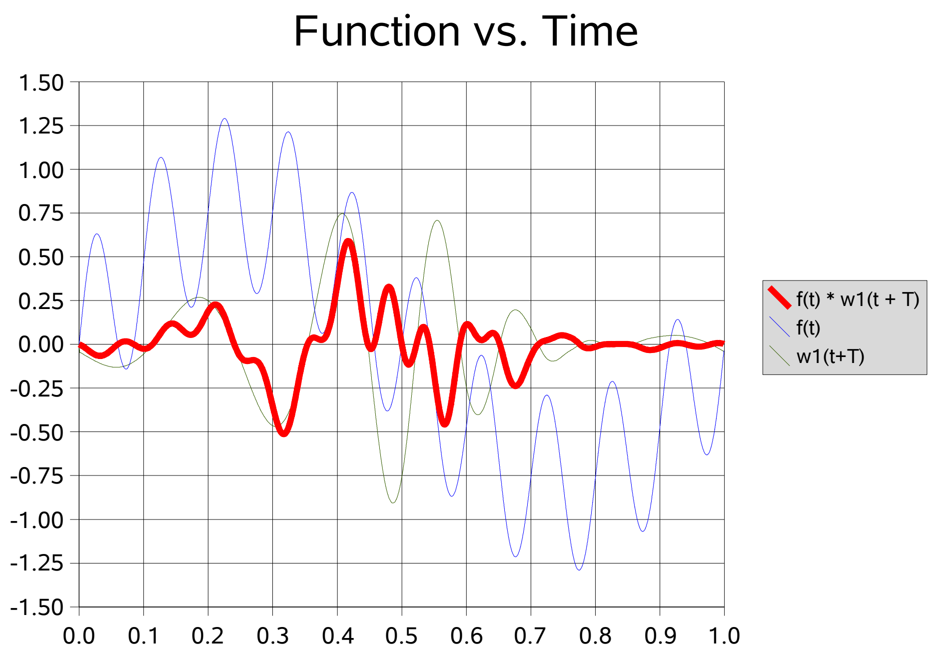
Wavelet Analysis



Wavelet Analysis



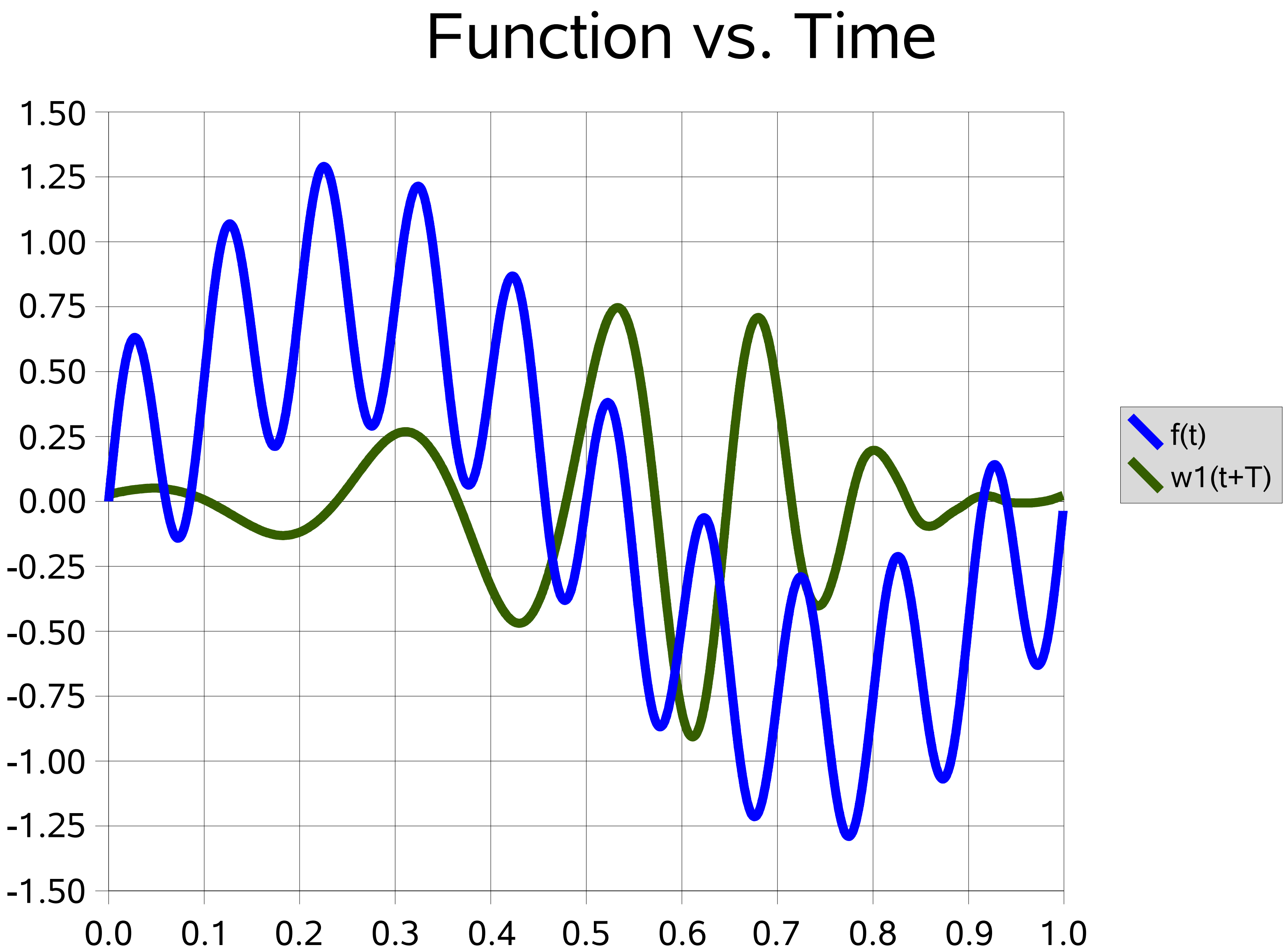
Wavelet Analysis



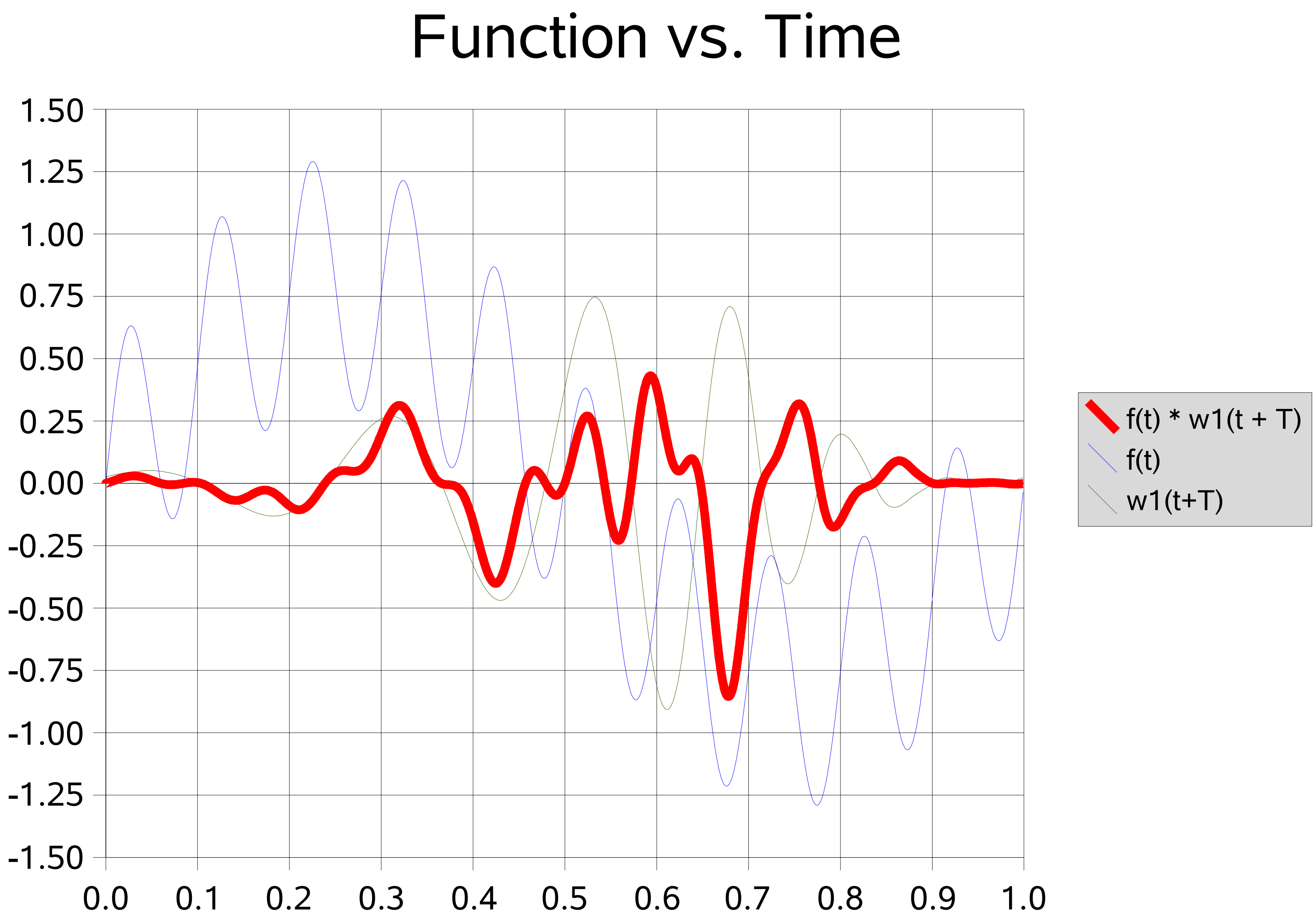
$$\gamma = \int_{-\infty}^{\infty} f(t)\Psi_{j,k}(t)\,dt$$

To find wavelet transform, integrate the function times the wavelet.

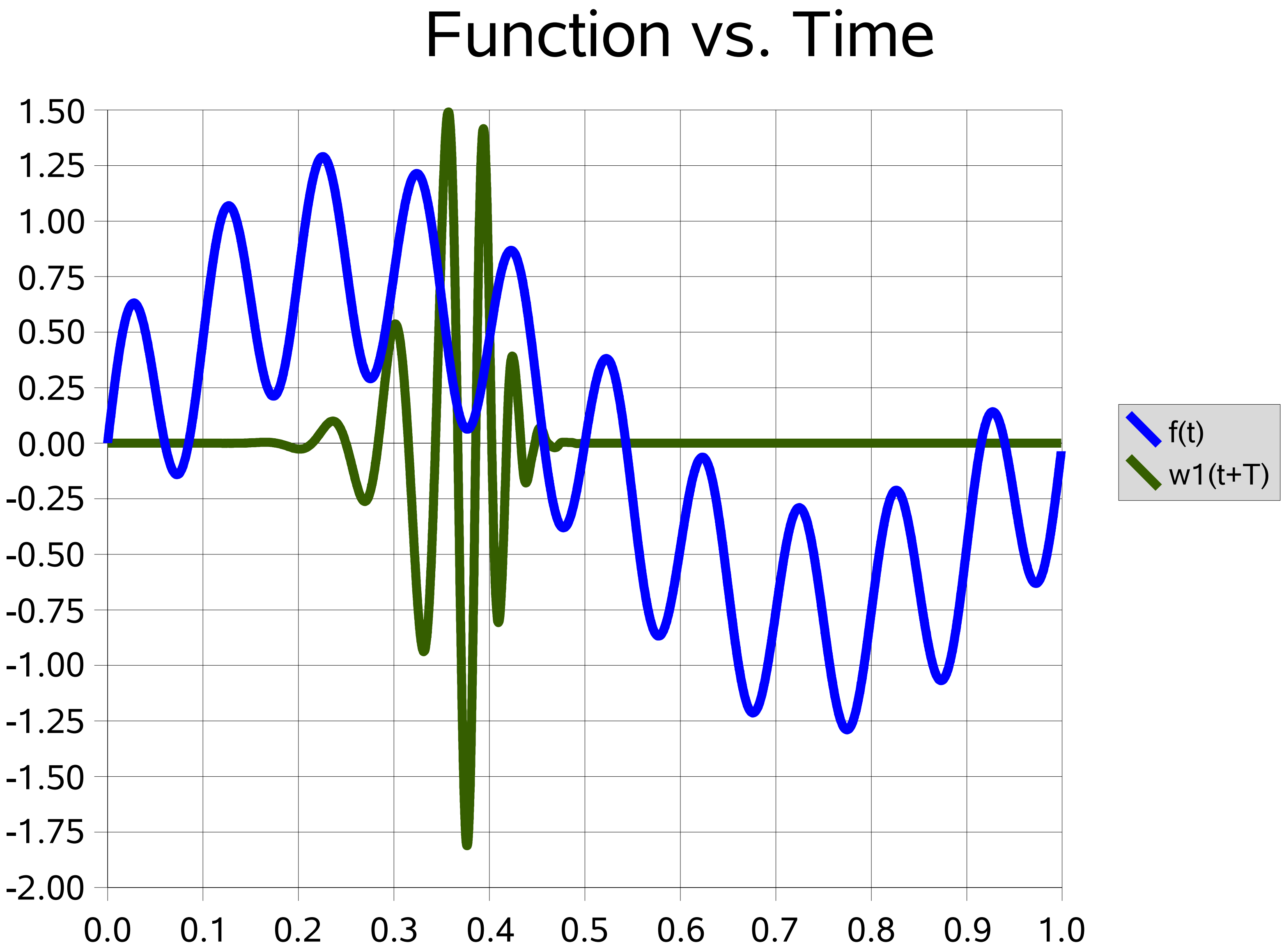
Wavelet Analysis



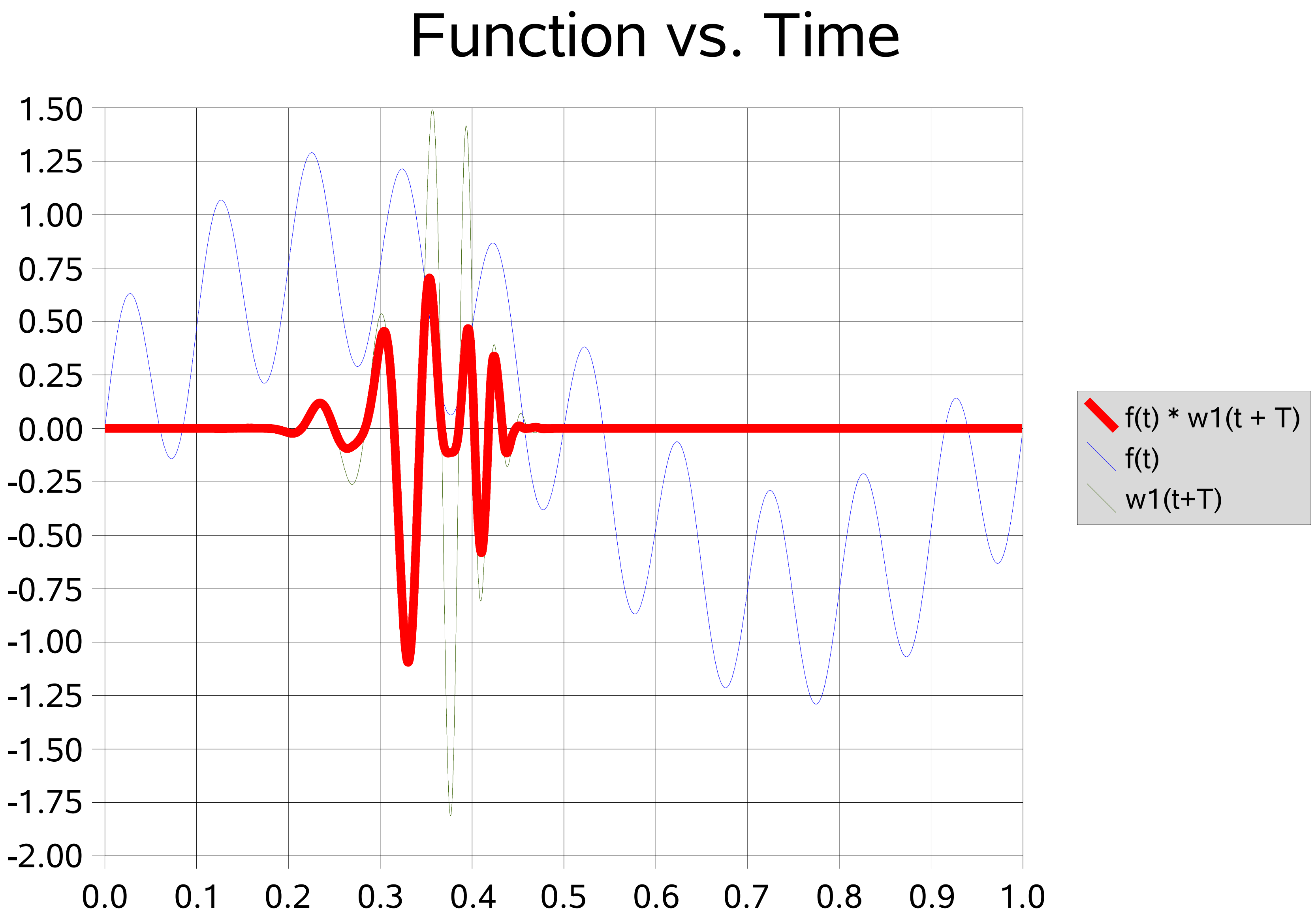
Wavelet Analysis

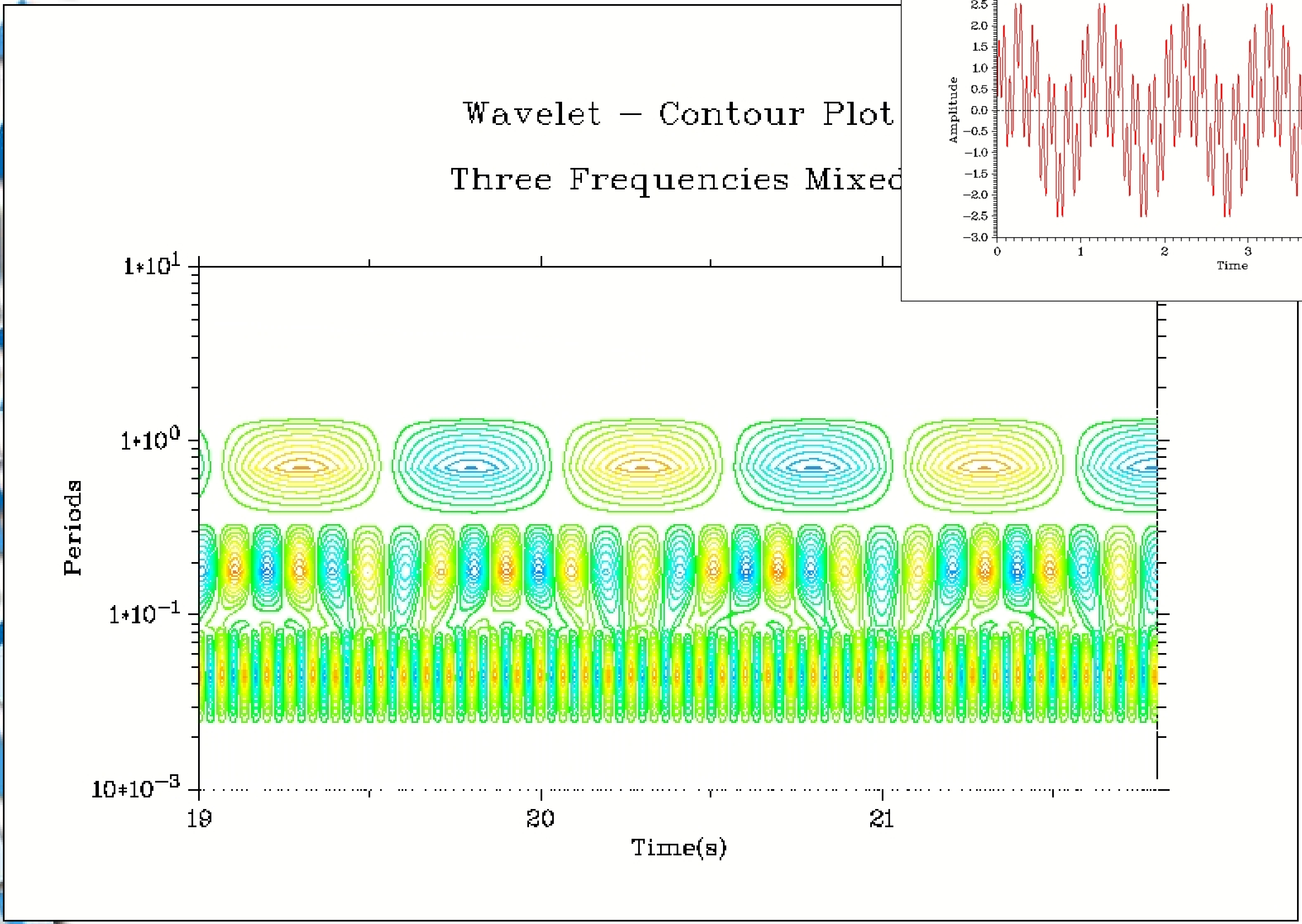


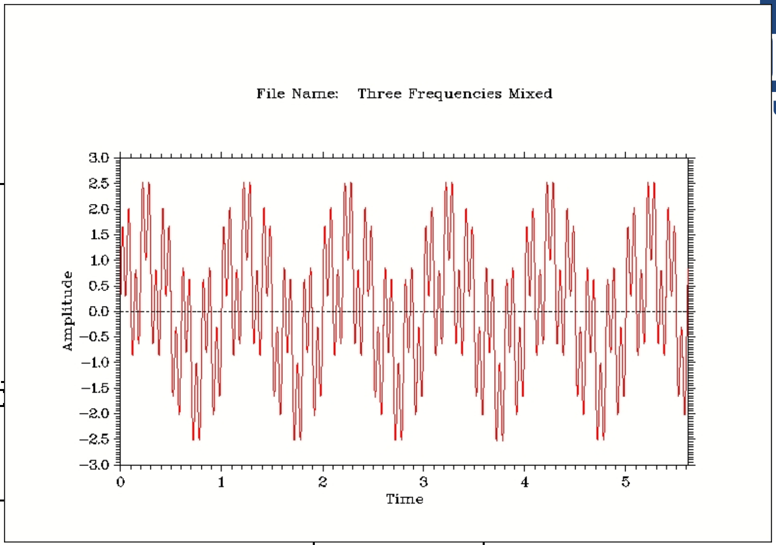
Wavelet Analysis

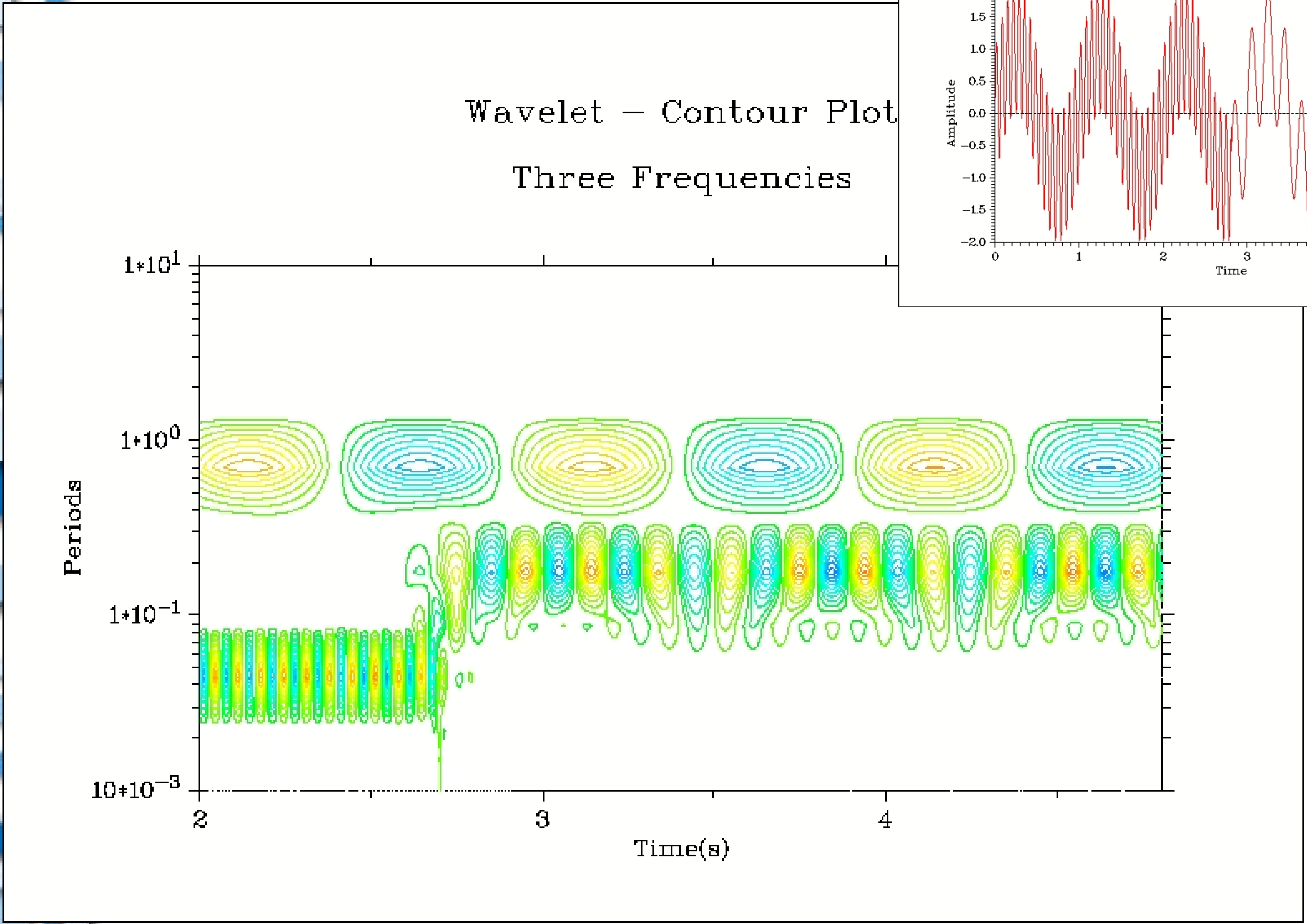


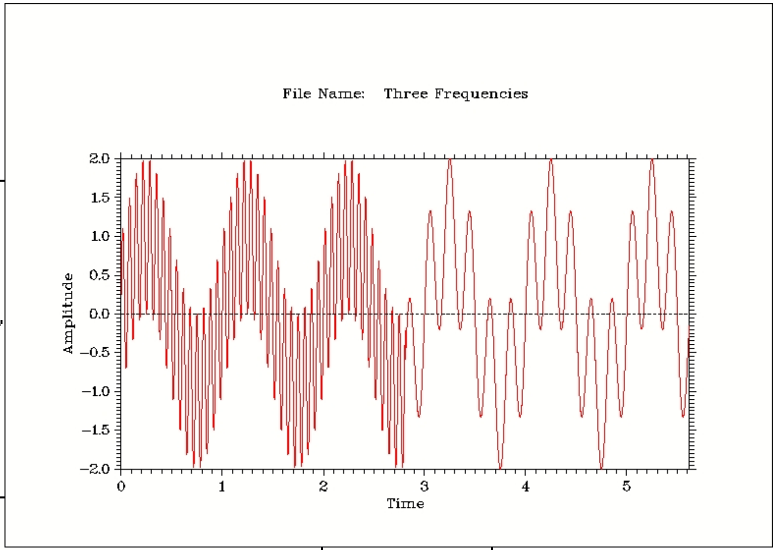
Wavelet Analysis




Wavelet – Contour Plot
Three Frequencies Mixed
Periods
Time(s)
1*10^1
1*10^0
1*10^-1
10*10^-3
19
20
21

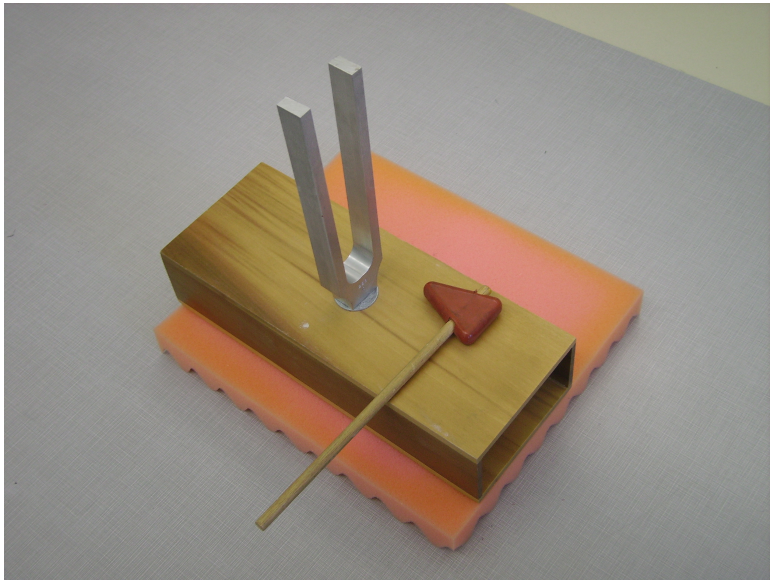

File Name: Three Frequencies Mixed
Amplitude
Time

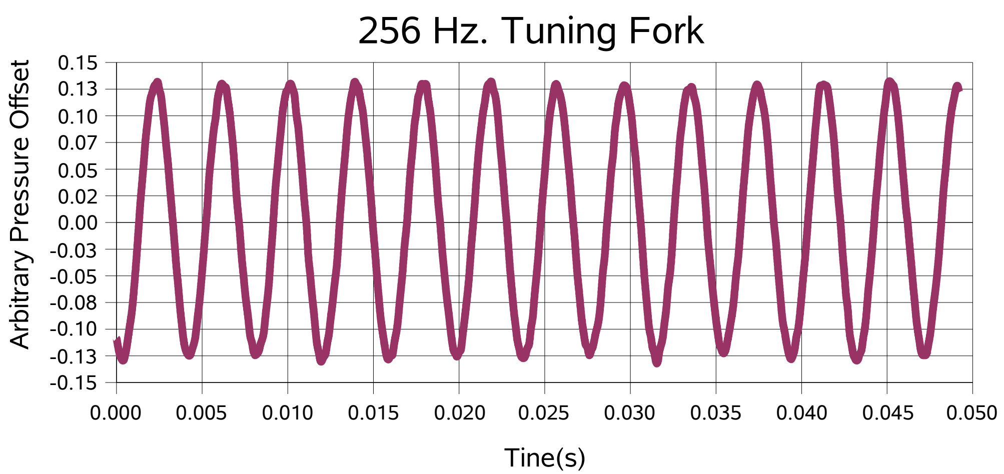


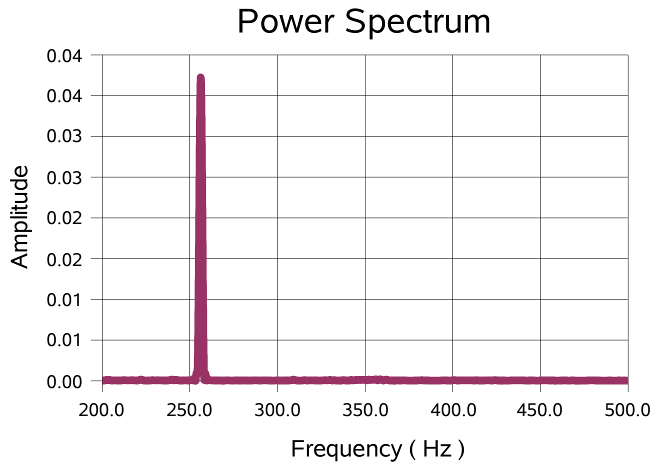


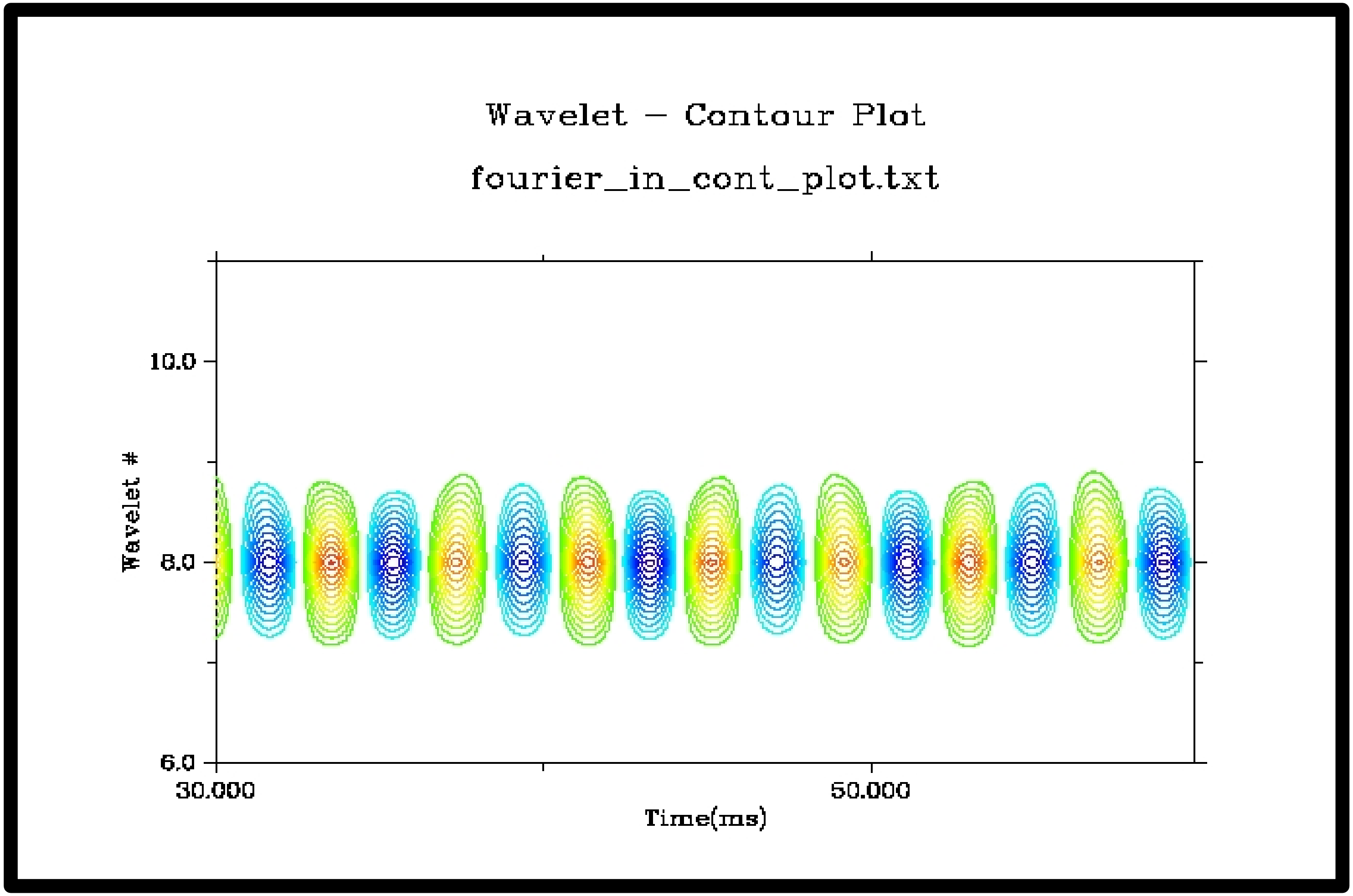
Drexel UNIVERSITY

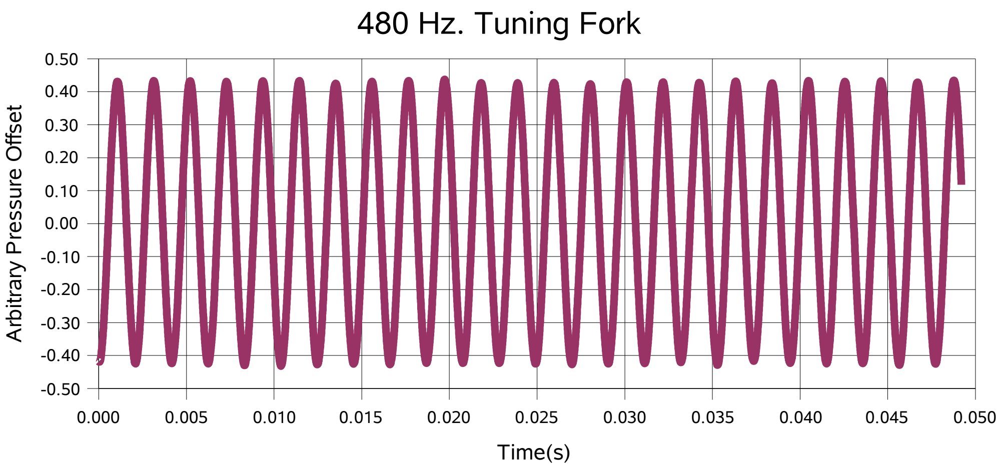
Tuning Forks

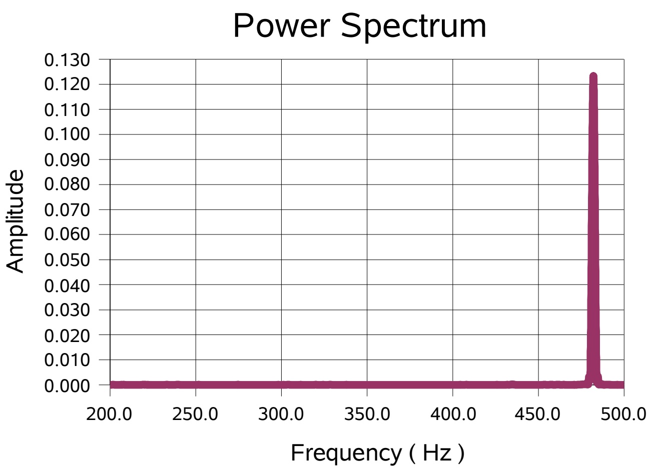


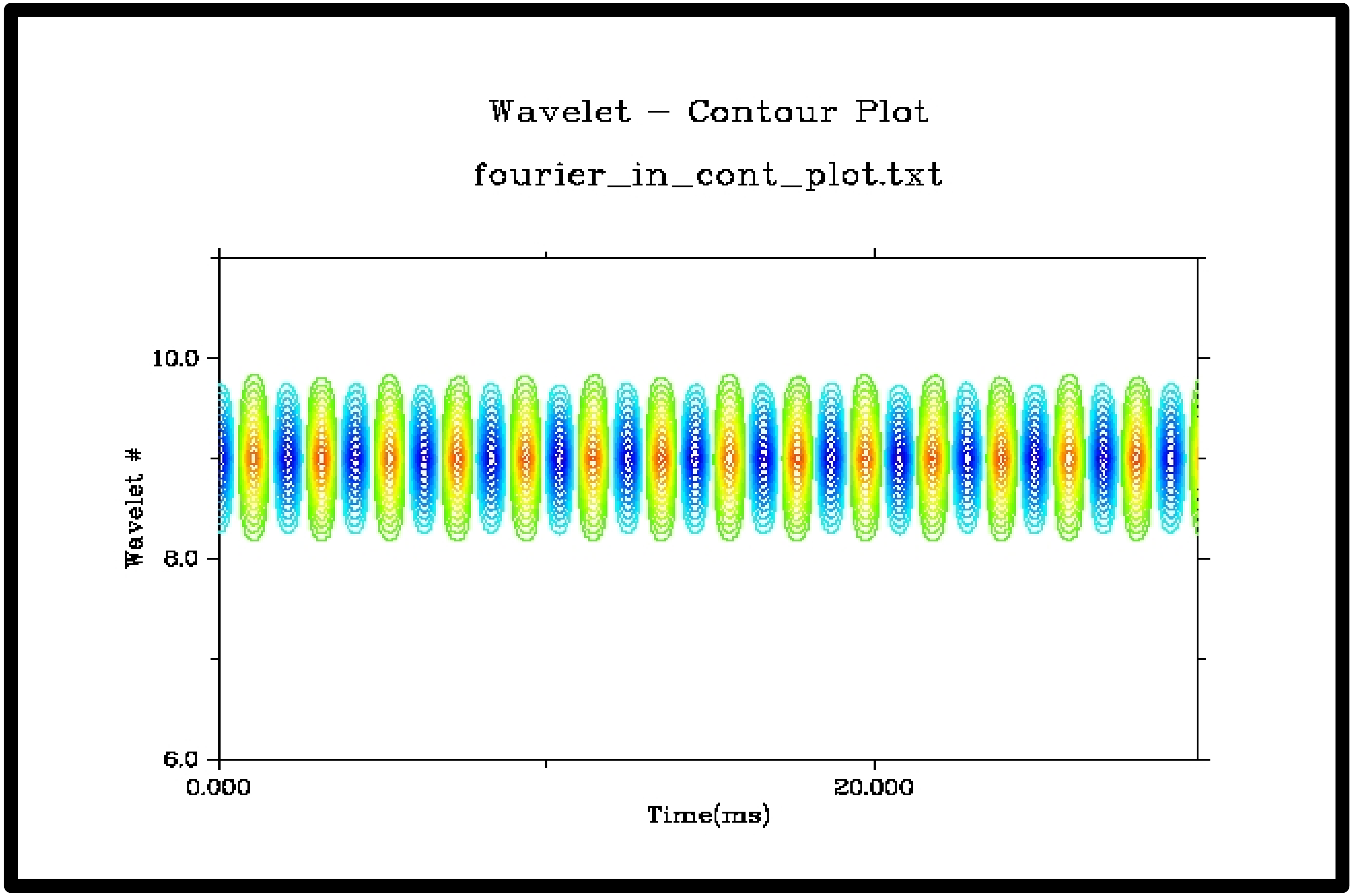

256 Hz. Tuning Fork
Arbitrary Pressure Offset
0.15
0.13
0.10
0.07
0.05
0.02
0.00
-0.03
-0.05
-0.08
-0.10
-0.13
-0.15
0.000
0.005
0.010
0.015
0.020
0.025
0.030
0.035
0.040
0.045
0.050
Tine(s)

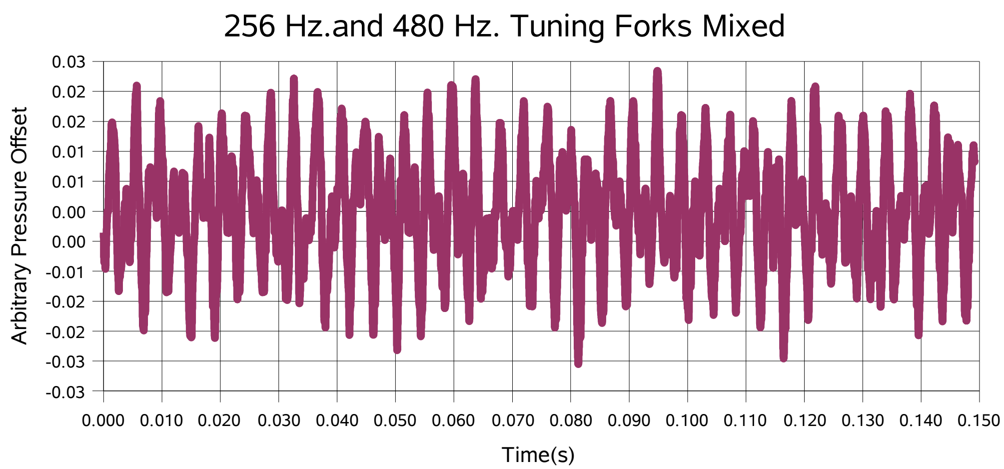

Power Spectrum
Amplitude
0.04
0.04
0.03
0.03
0.02
0.02
0.01
0.01
0.00
200.0
250.0
300.0
350.0
400.0
450.0
500.0
Frequency (Hz)

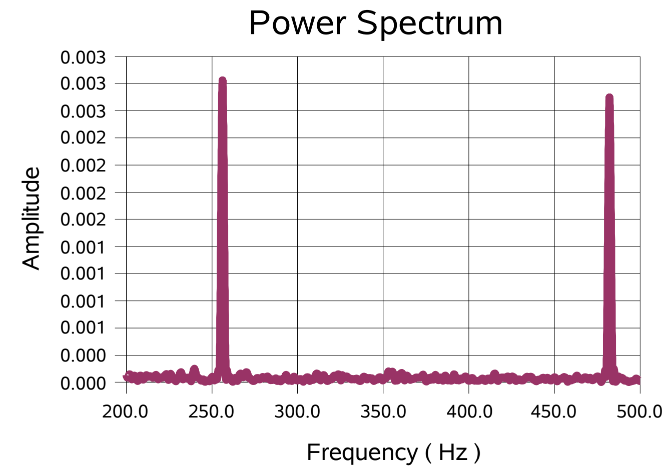

Wavelet – Contour Plot
fourier_in_cont_plot.txt
Wavelet #
10.0
8.0
6.0
30.000
50.000
Time(ms)

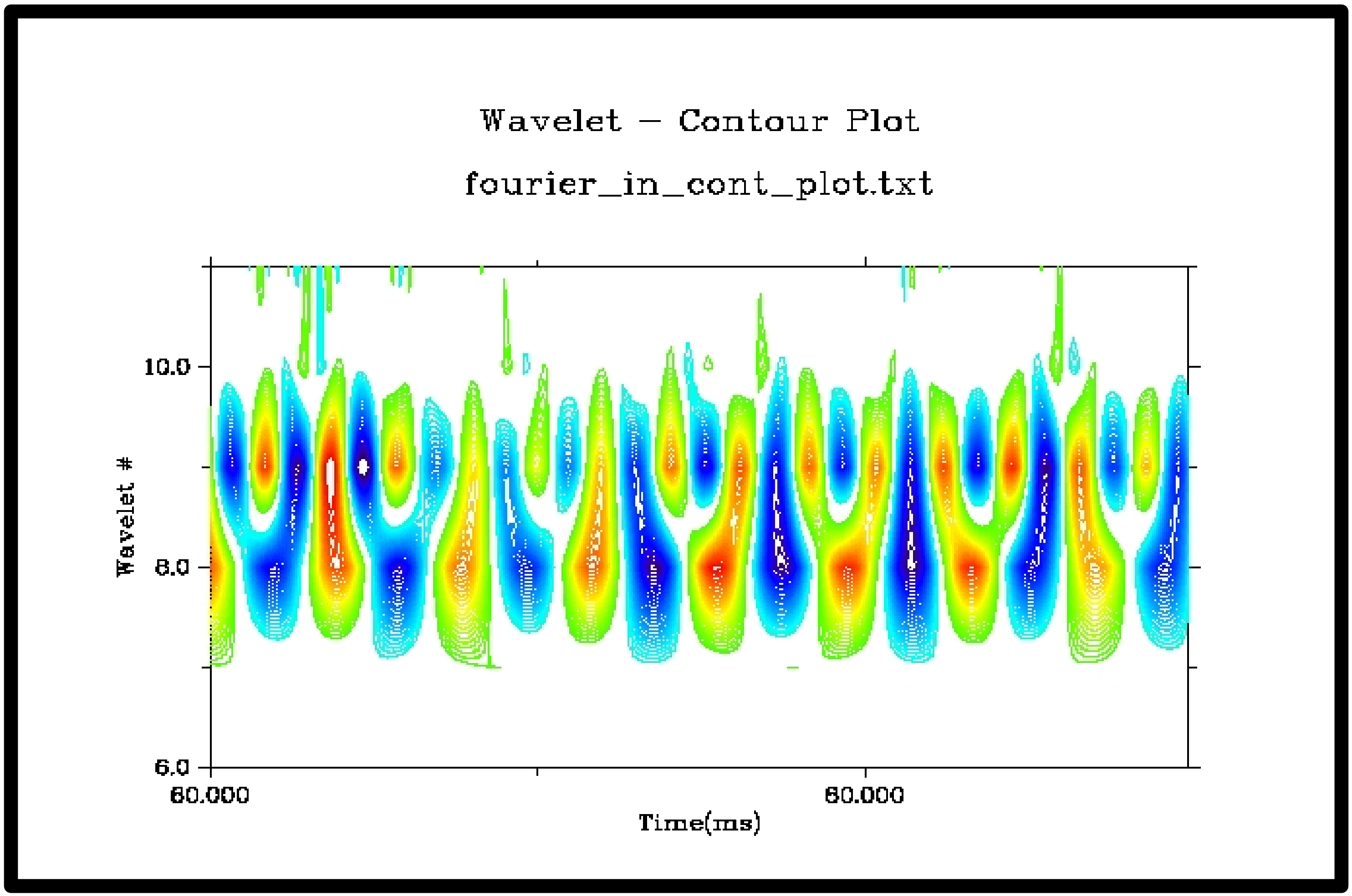

480 Hz. Tuning Fork
Arbitrary Pressure Offset
Time(s)
0.50
0.40
0.30
0.20
0.10
0.00
-0.10
-0.20
-0.30
-0.40
-0.50
0.000
0.005
0.010
0.015
0.020
0.025
0.030
0.035
0.040
0.045
0.050

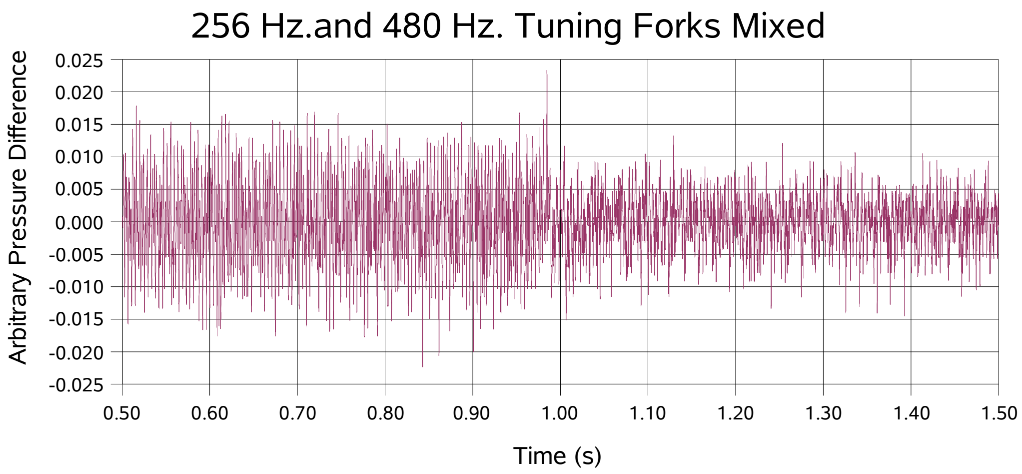

Power Spectrum
Amplitude
Frequency (Hz)
0.130
0.120
0.110
0.100
0.090
0.080
0.070
0.060
0.050
0.040
0.030
0.020
0.010
0.000
200.0
250.0
300.0
350.0
400.0
450.0
500.0

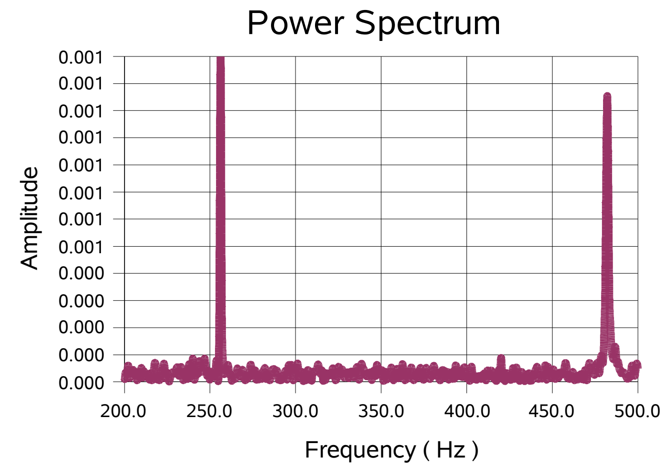

Wavelet – Contour Plot
fourier_in_cont_plot.txt
Wavelet #
Time(ms)
10.0
8.0
6.0
0.000
20.000

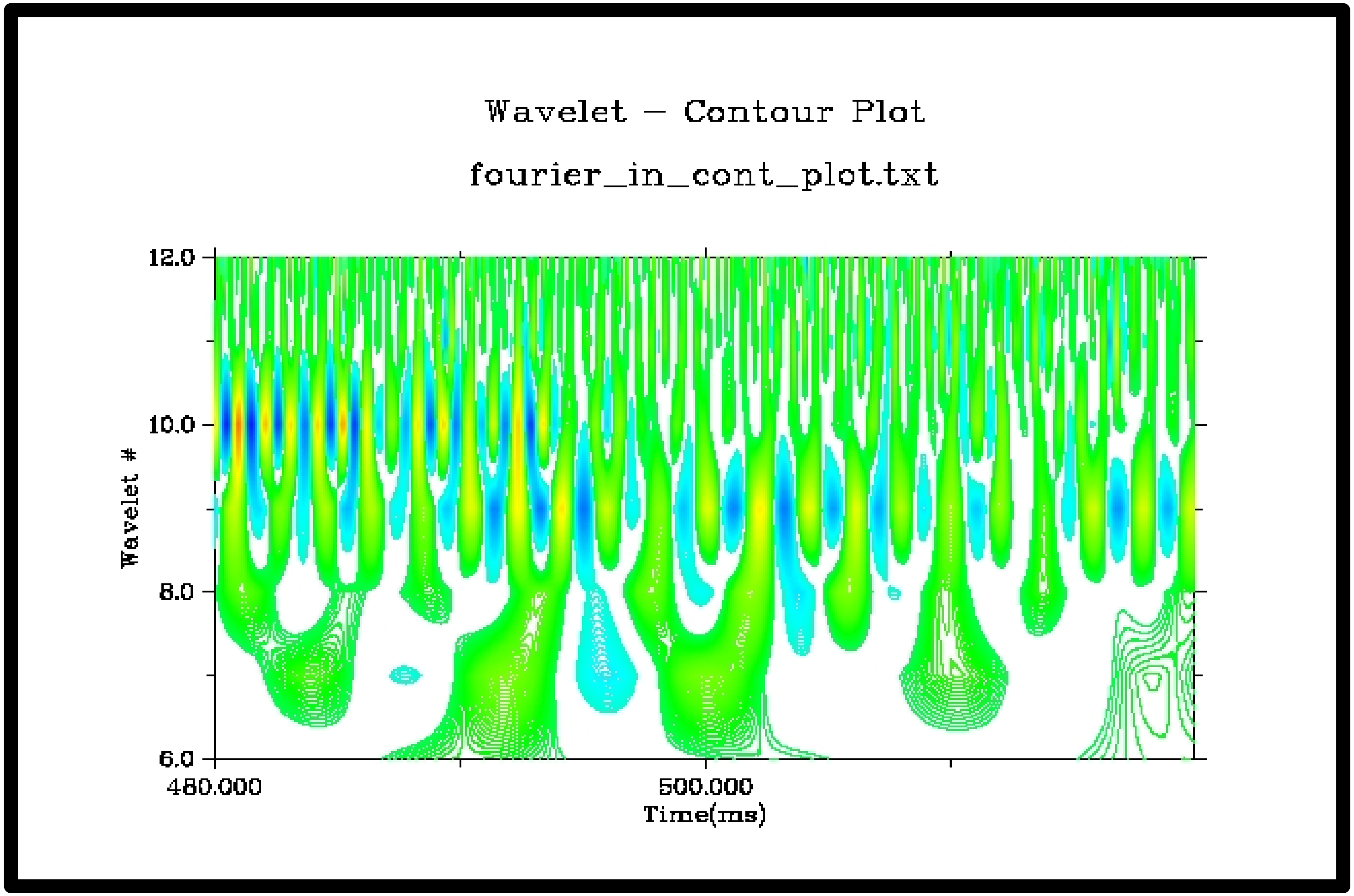

256 Hz.and 480 Hz. Tuning Forks Mixed
Arbitrary Pressure Offset
Time(s)

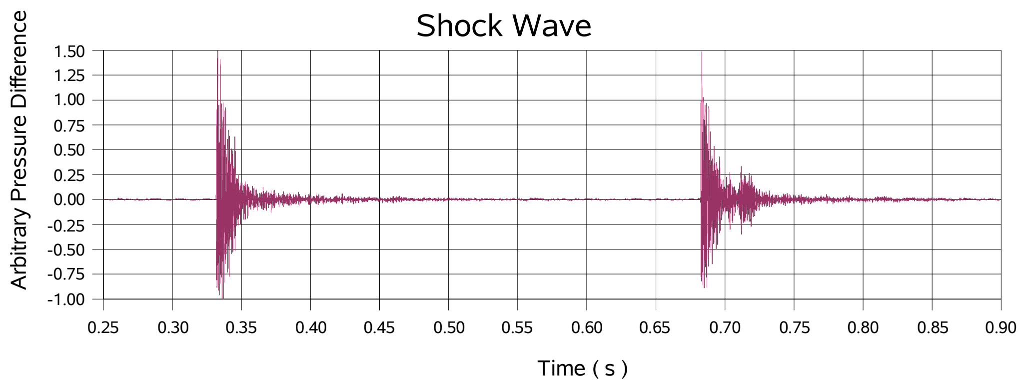

Power Spectrum
Amplitude
Frequency (Hz)

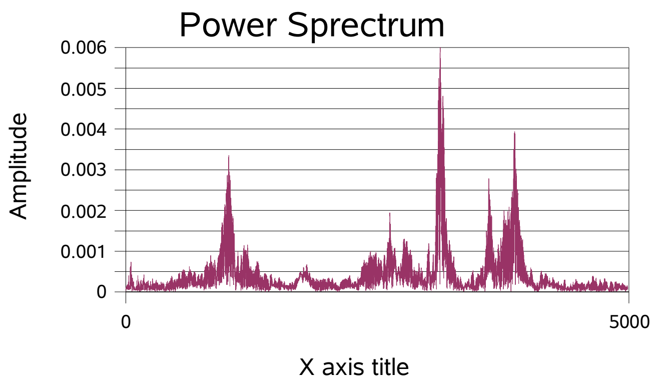

Wavelet – Contour Plot
fourier_in_cont_plot.txt
Wavelet #
Time(ms)

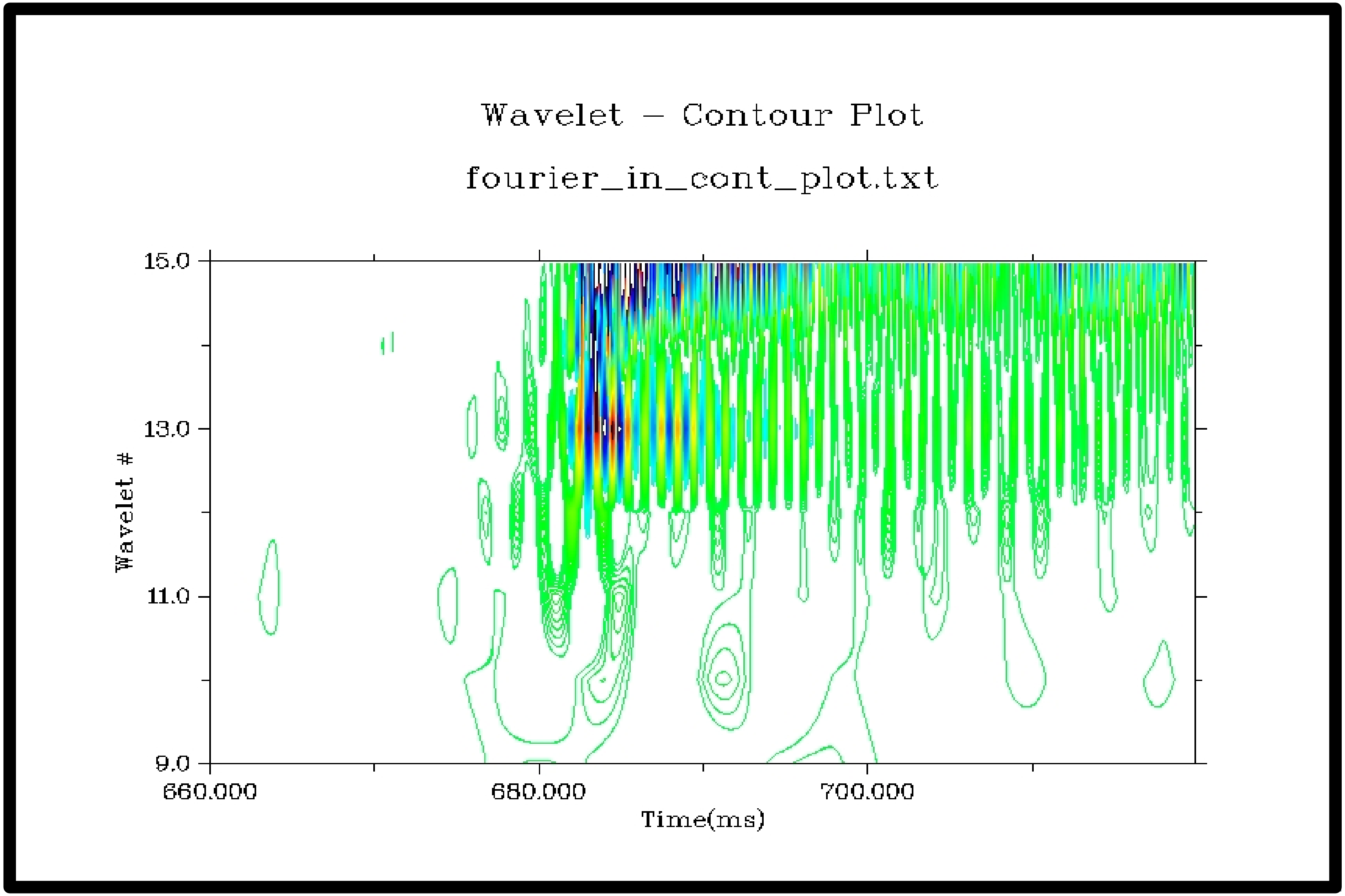

256 Hz.and 480 Hz. Tuning Forks Mixed
Arbitrary Pressure Difference
0.025
0.020
0.015
0.010
0.005
0.000
-0.005
-0.010
-0.015
-0.020
-0.025
0.50
0.60
0.70
0.80
0.90
1.00
1.10
1.20
1.30
1.40
1.50
Time (s)


Power Spectrum
Amplitude
0.001
0.000
200.0
250.0
300.0
350.0
400.0
450.0
500.0
Frequency (Hz)


Wavelet - Contour Plot
fourier_in_cont_plot.txt
Wavelet #
12.0
10.0
8.0
6.0
480.000
500.000
Time(ms)


Shock Wave
Arbitrary Pressure Difference
1.50
1.25
1.00
0.75
0.50
0.25
0.00
-0.25
-0.50
-0.75
-1.00
0.25 0.30 0.35 0.40 0.45 0.50 0.55 0.60 0.65 0.70 0.75 0.80 0.85 0.90
Time (s)


Power Sprectrum
Amplitude
0.006
0.005
0.004
0.003
0.002
0.001
0
0
5000
X axis title

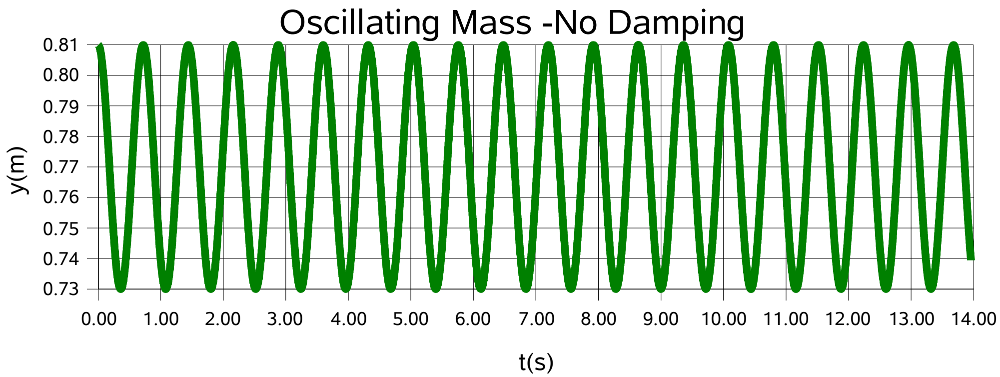

Wavelet – Contour Plot
fourier_in_cont_plot.txt
Wavelet #
15.0
13.0
11.0
9.0
660.000
680.000
700.000
Time(ms)

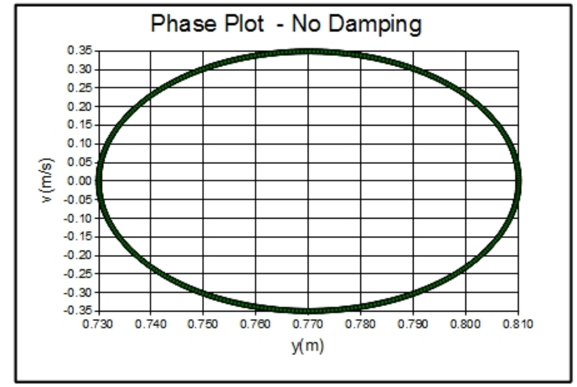
4.) Other Applications

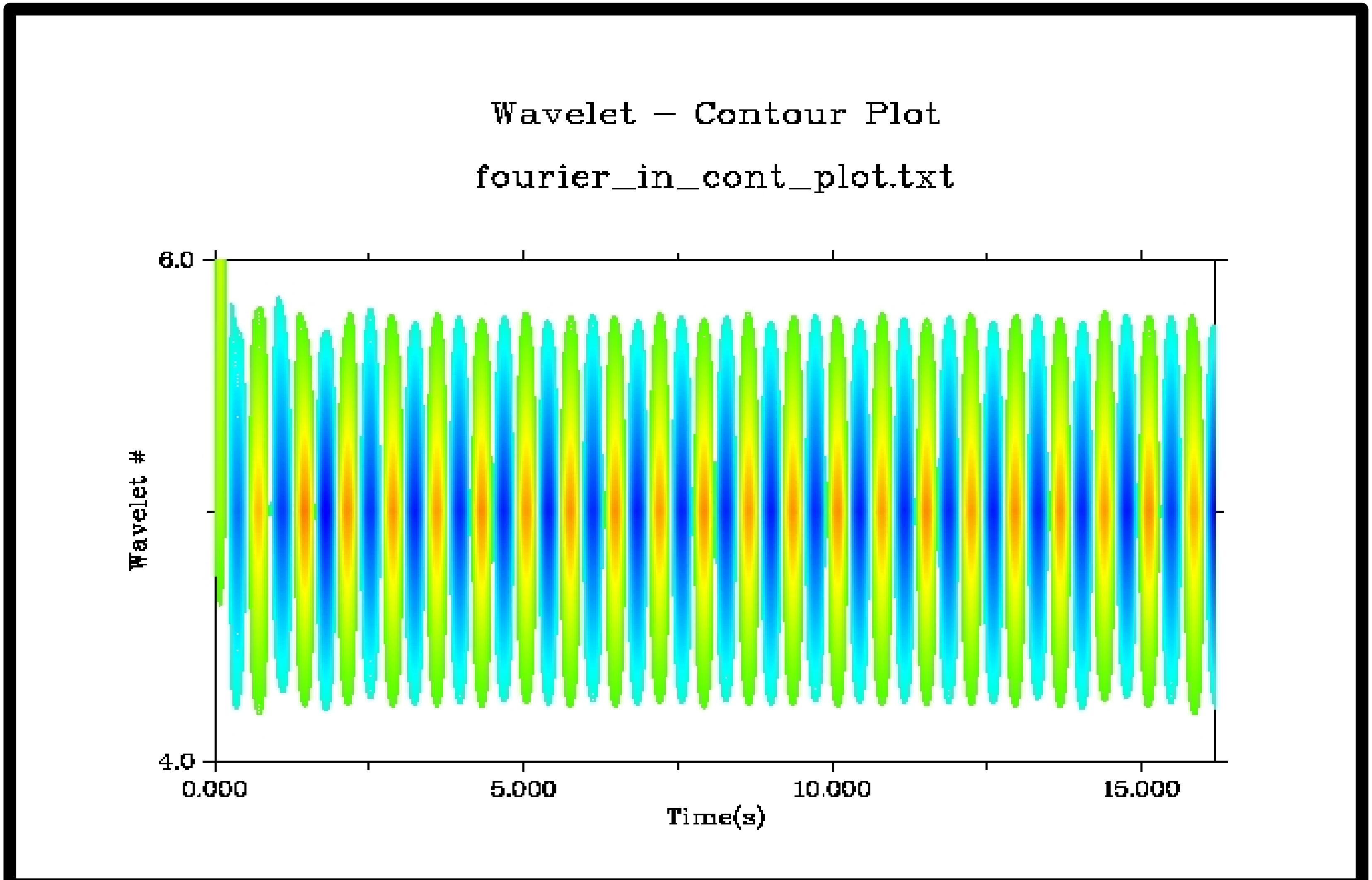
a)Mass on a Spring

b)Temperature Data

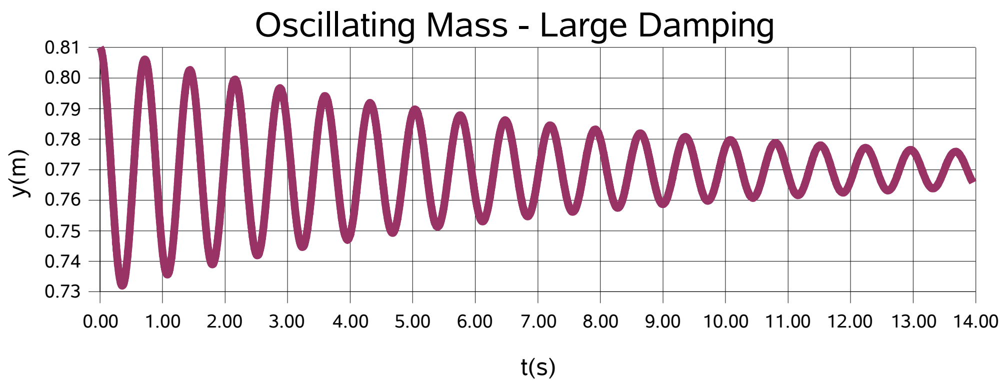
Mass on a Spring

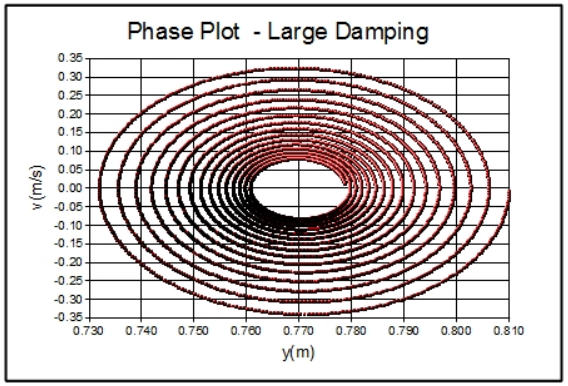


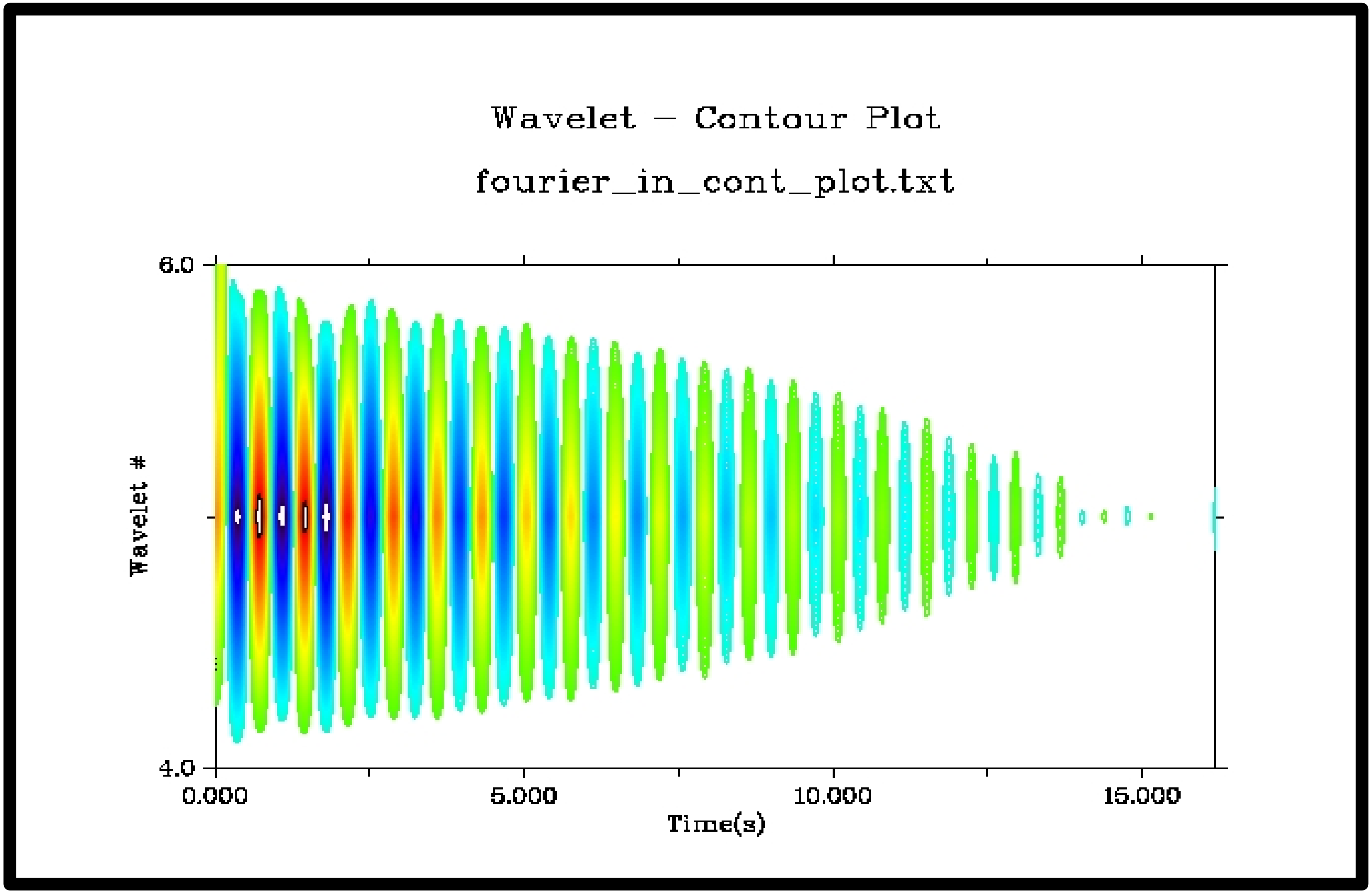




Simulated Data



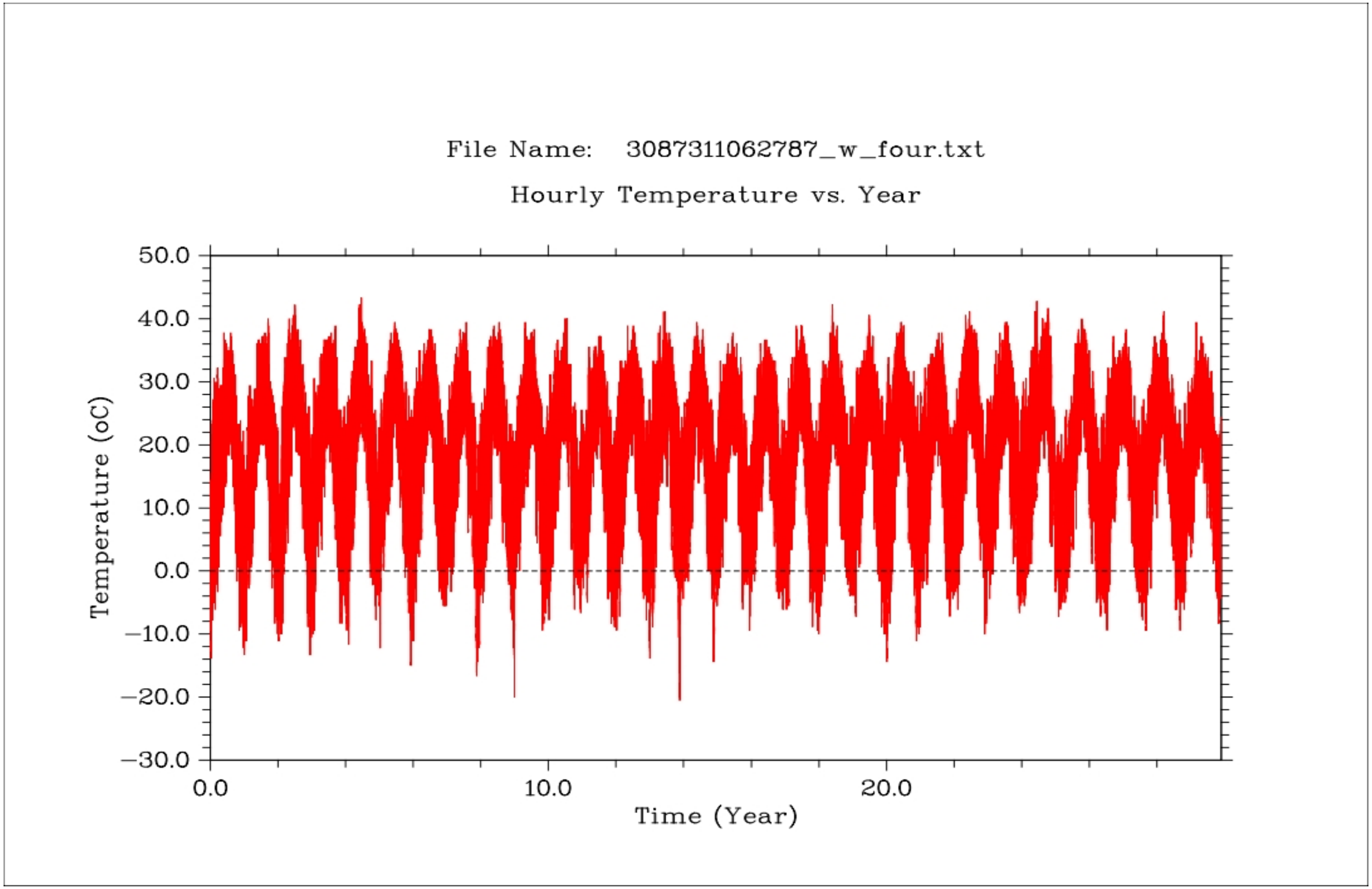




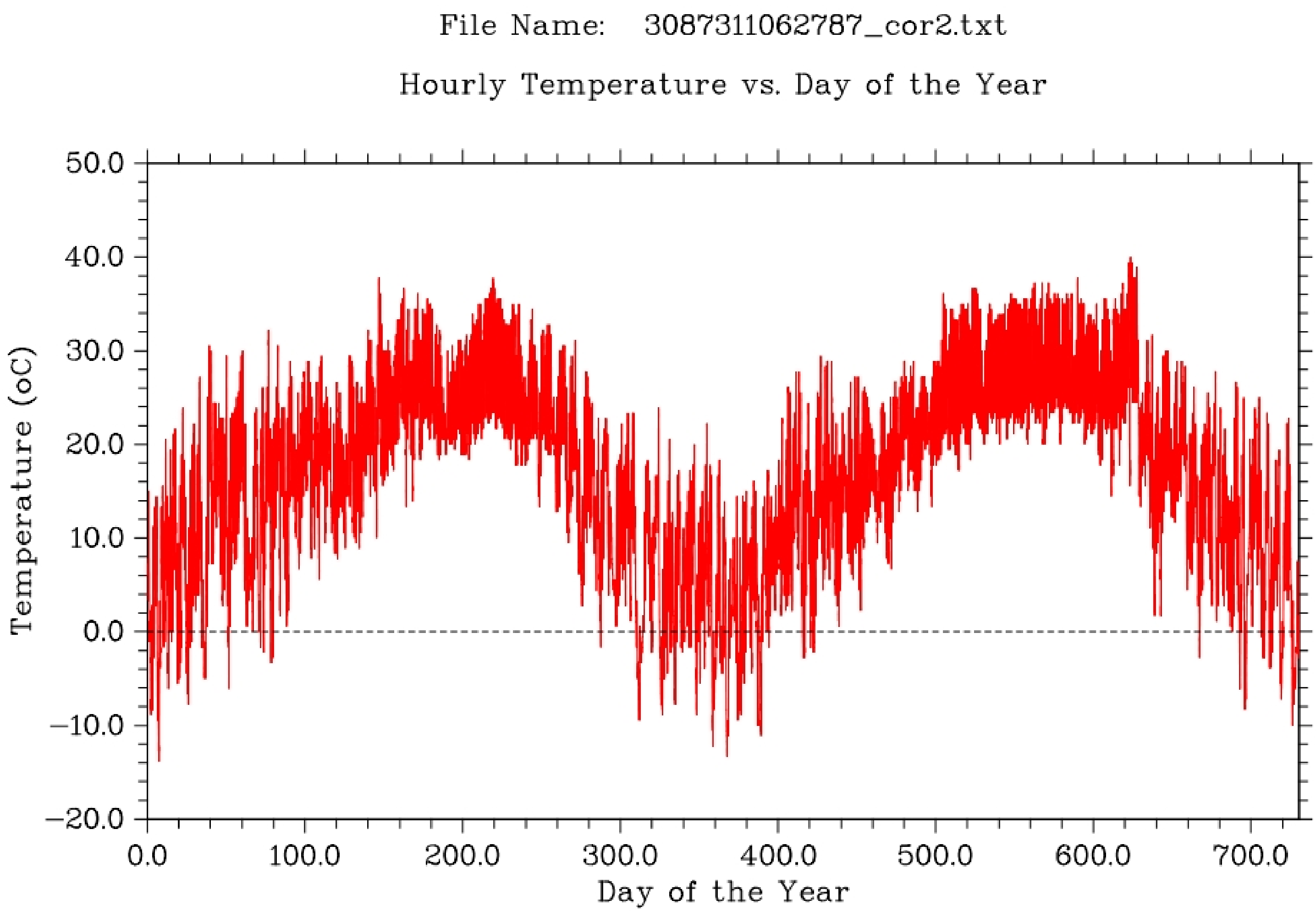
Simulated Data

Temperature Data

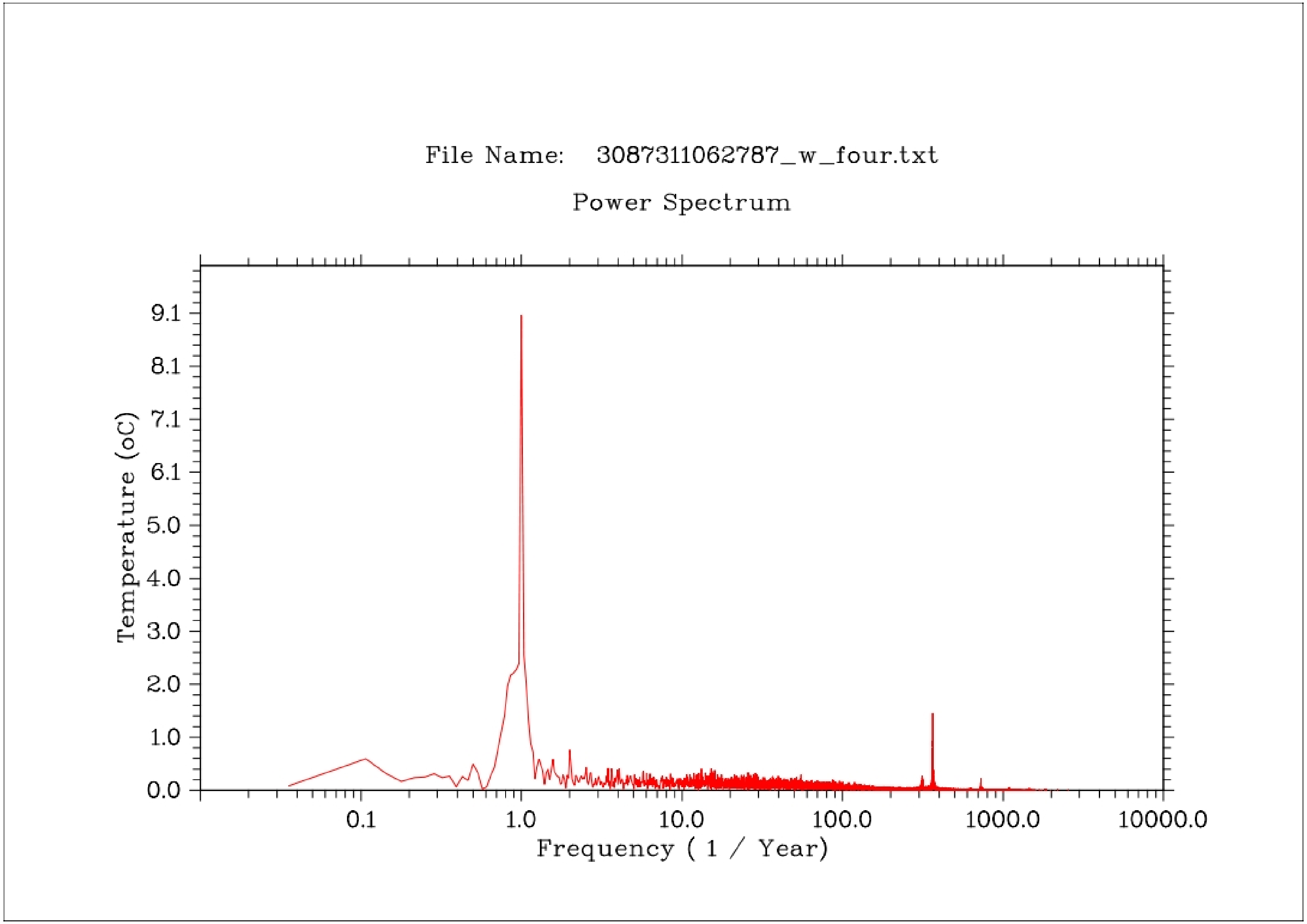
We can also use Fourier Analysis to find out what frequencies make up a Wave or Signal.



Hourly Average Temperature



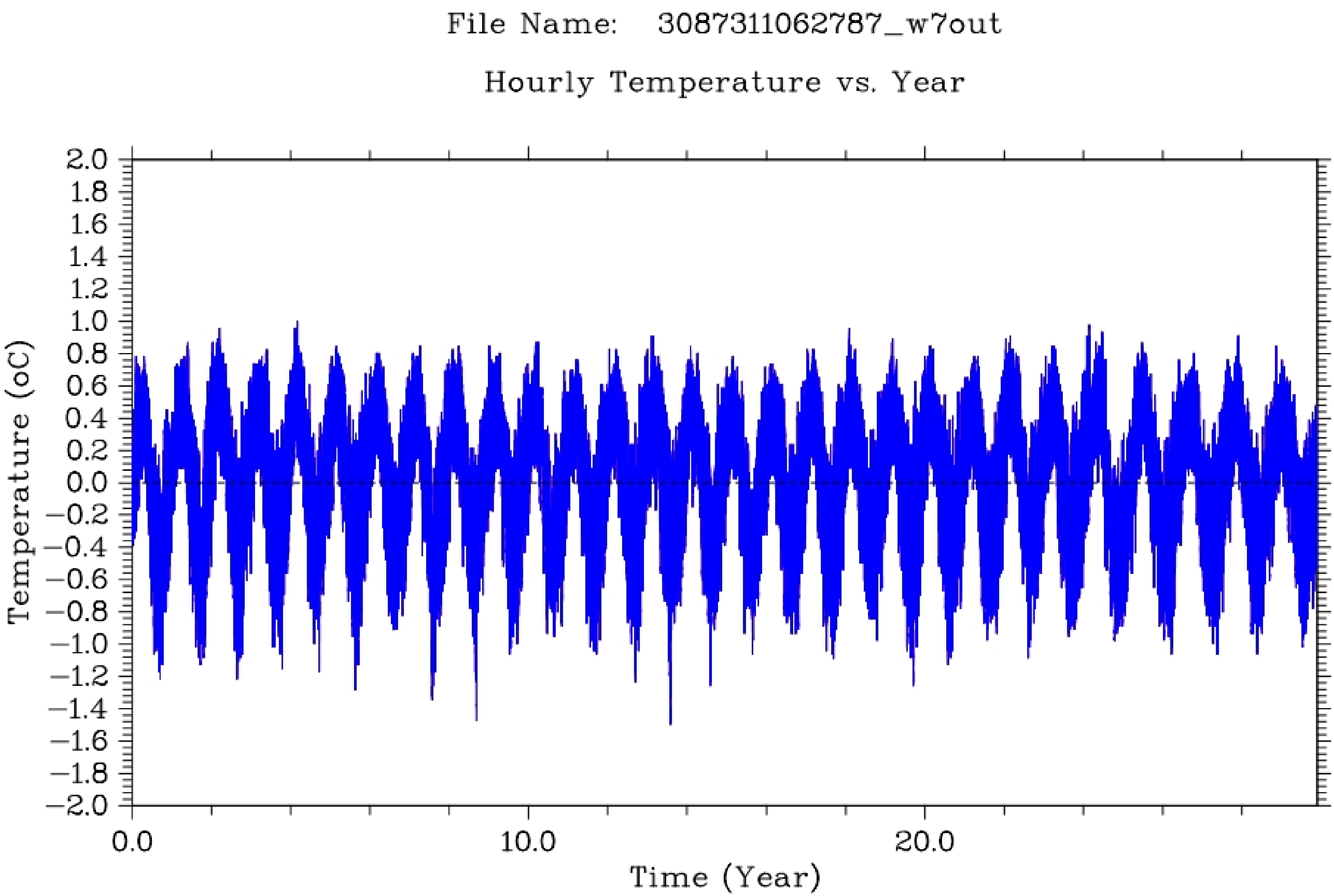
Temperature – Power Spectrum



Temperature – Power Spectrum

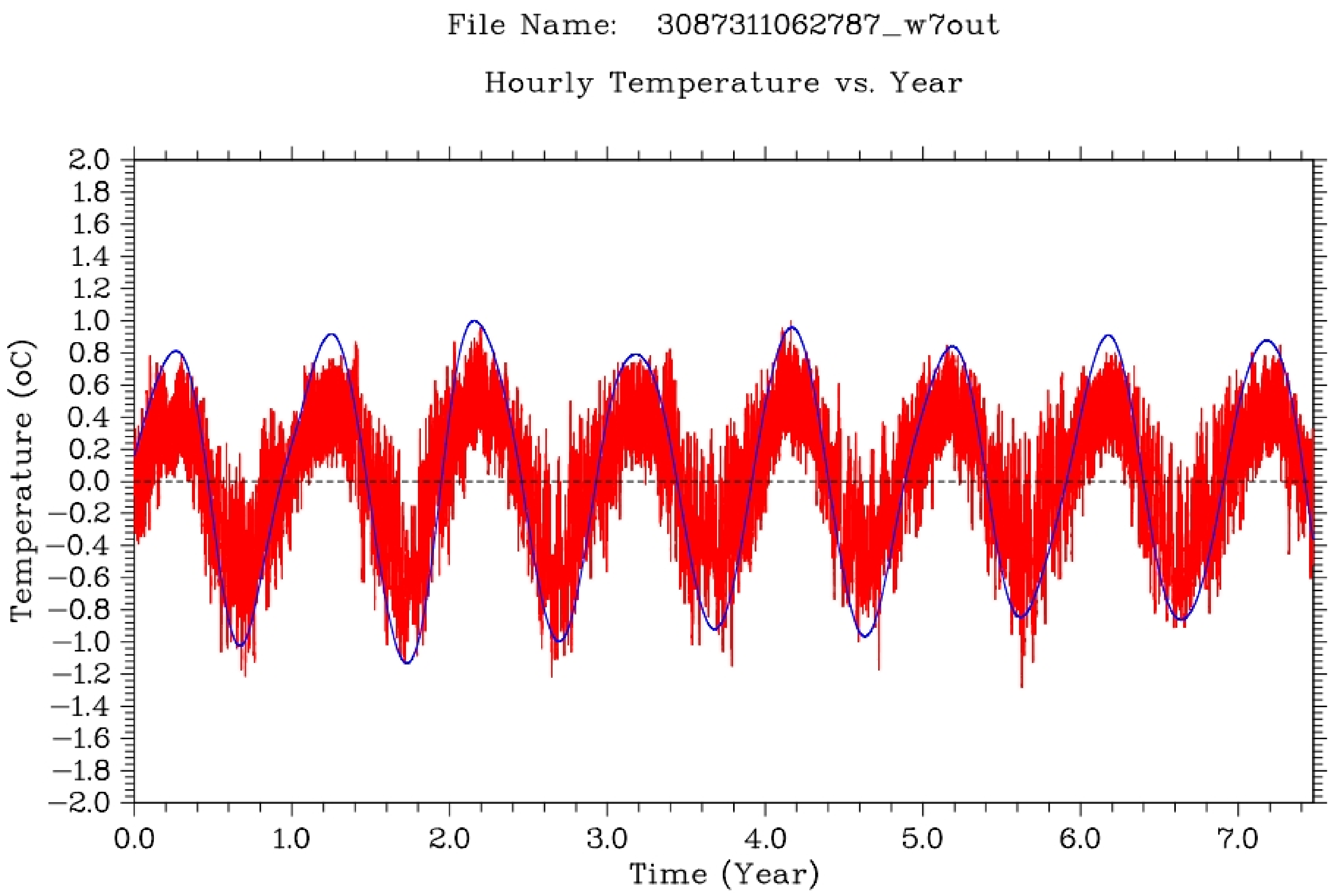
azero = 17.724915
i= 21 f= 0.750000 a= -0.472239 b= 1.007747 A= 1.112908
i= 22 f= 0.785714 a= -0.073951 b= 1.434289 A= 1.436194
i= 23 f= 0.821429 a= 0.190072 b= 1.979196 A= 1.988302
i= 24 f= 0.857143 a= 1.106927 b= 1.899358 A= 2.198374
i= 25 f= 0.892857 a= 1.640487 b= 1.510557 A= 2.230018
i= 26 f= 0.928571 a= 2.265698 b= 0.372863 A= 2.296174
i= 27 f= 0.964286 a= 2.217068 b= -0.963278 A= 2.417291
i= 28 f= 1.000000 a= -8.963787 b= 1.290257 A= 9.056171
i= 29 f= 1.035714 a= 0.669938 b= -2.489800 A= 2.578356
i= 30 f= 1.071429 a= 0.745567 b= -1.862980 A= 2.006630
i= 31 f= 1.107143 a= 0.381943 b= -1.296685 A= 1.351766
i= 32 f= 1.142857 a= 0.065296 b= -0.882051 A= 0.884465
i= 10215 f= 364.821442 a= 0.222982 b= 0.833274 A= 0.862593
i= 10216 f= 364.857147 a= 0.079152 b= -0.833938 A= 0.837686
i= 10217 f= 364.892853 a= -0.447426 b= 1.263199 A= 1.340098
i= 10218 f= 364.928589 a= -0.205658 b= -1.456414 A= 1.470863
i= 10219 f= 364.964294 a= 0.776877 b= 0.367377 A= 0.859362
i= 10220 f= 365.000000 a= -0.343463 b= 0.867783 A= 0.933281

Wavelet Analysis for Temperature Data



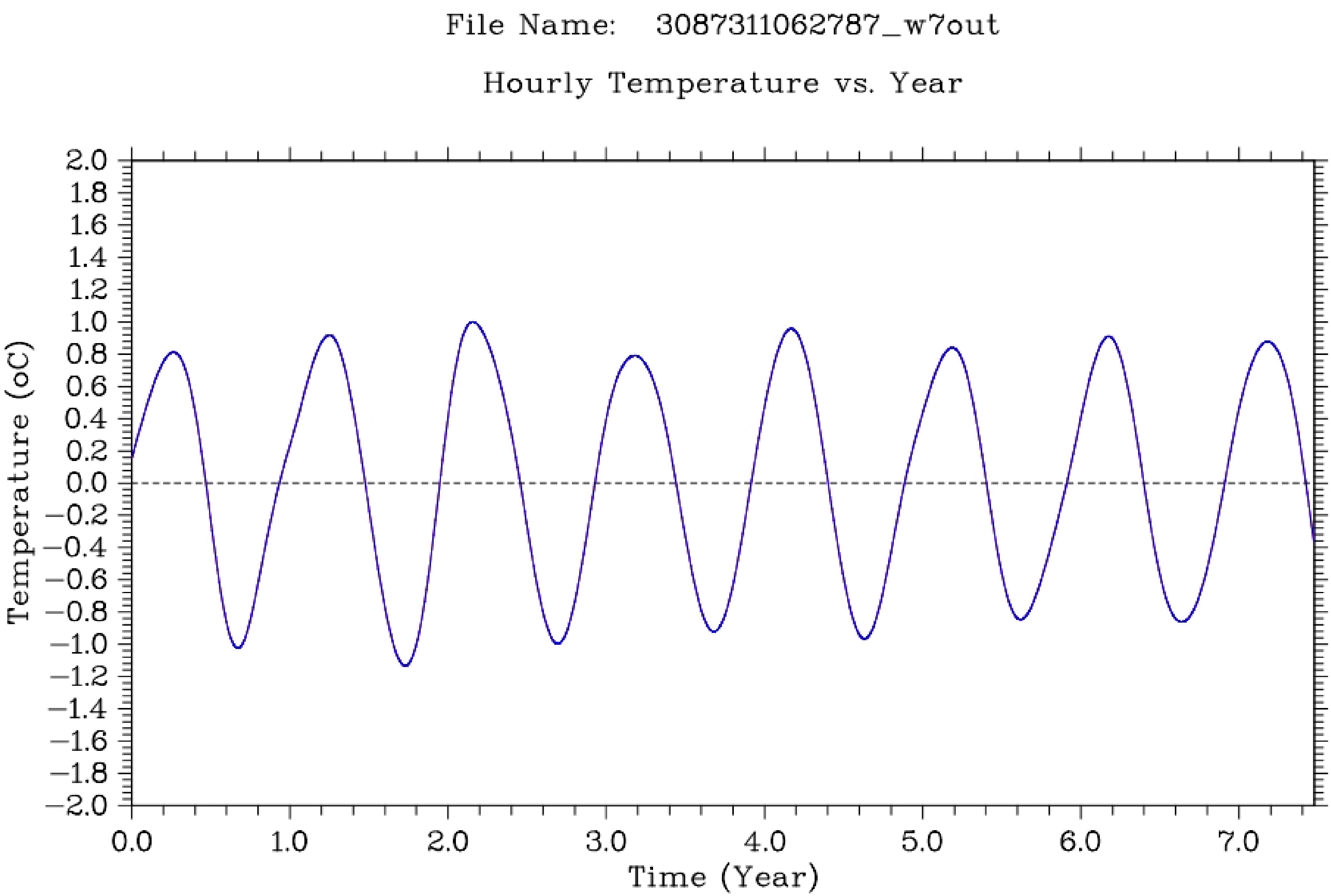
Original Signal

Wavelet Analysis for Temperature Data



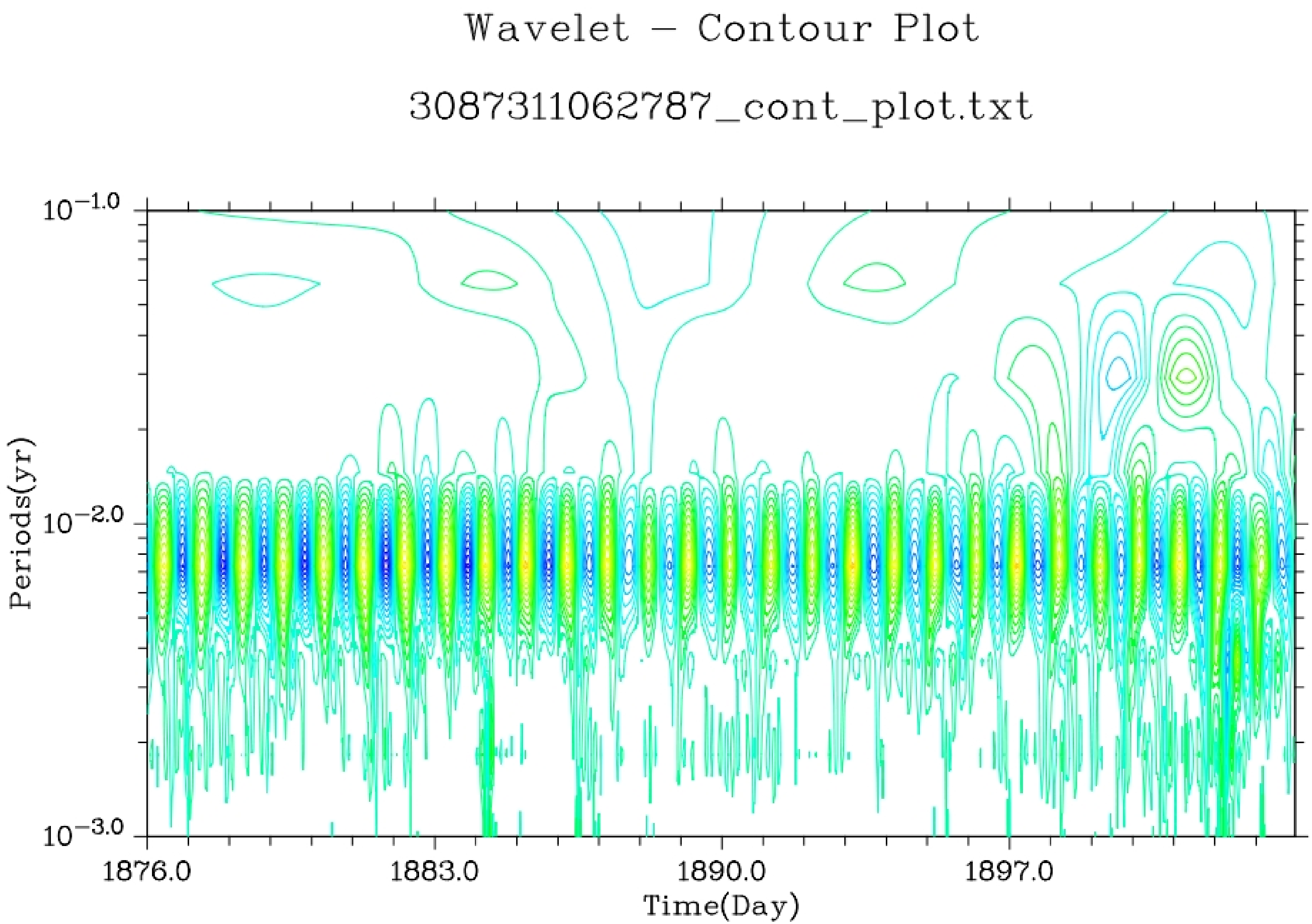
Original and Filtered Signal

Wavelet Analysis for Temperature Data

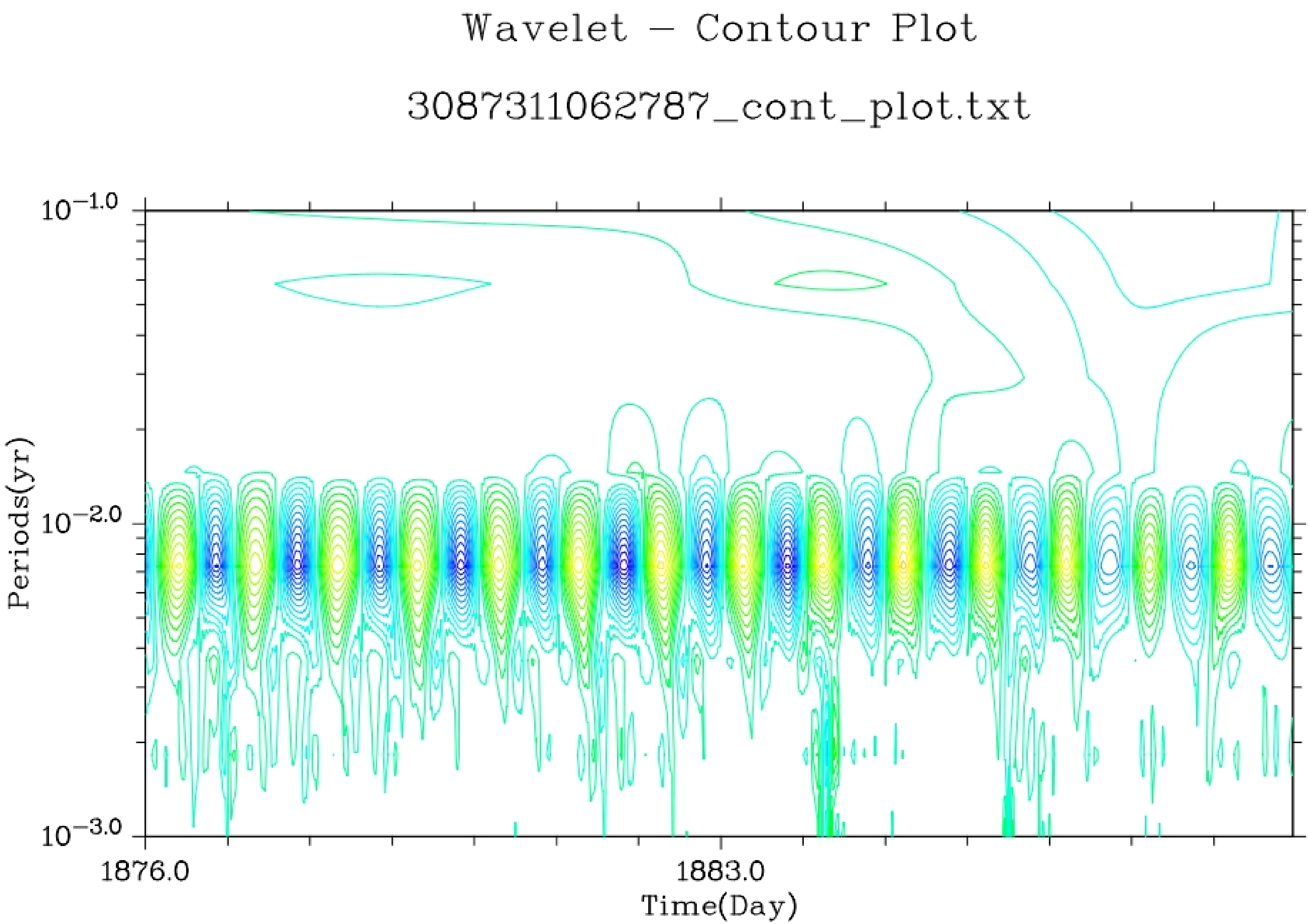


Filtered Signal

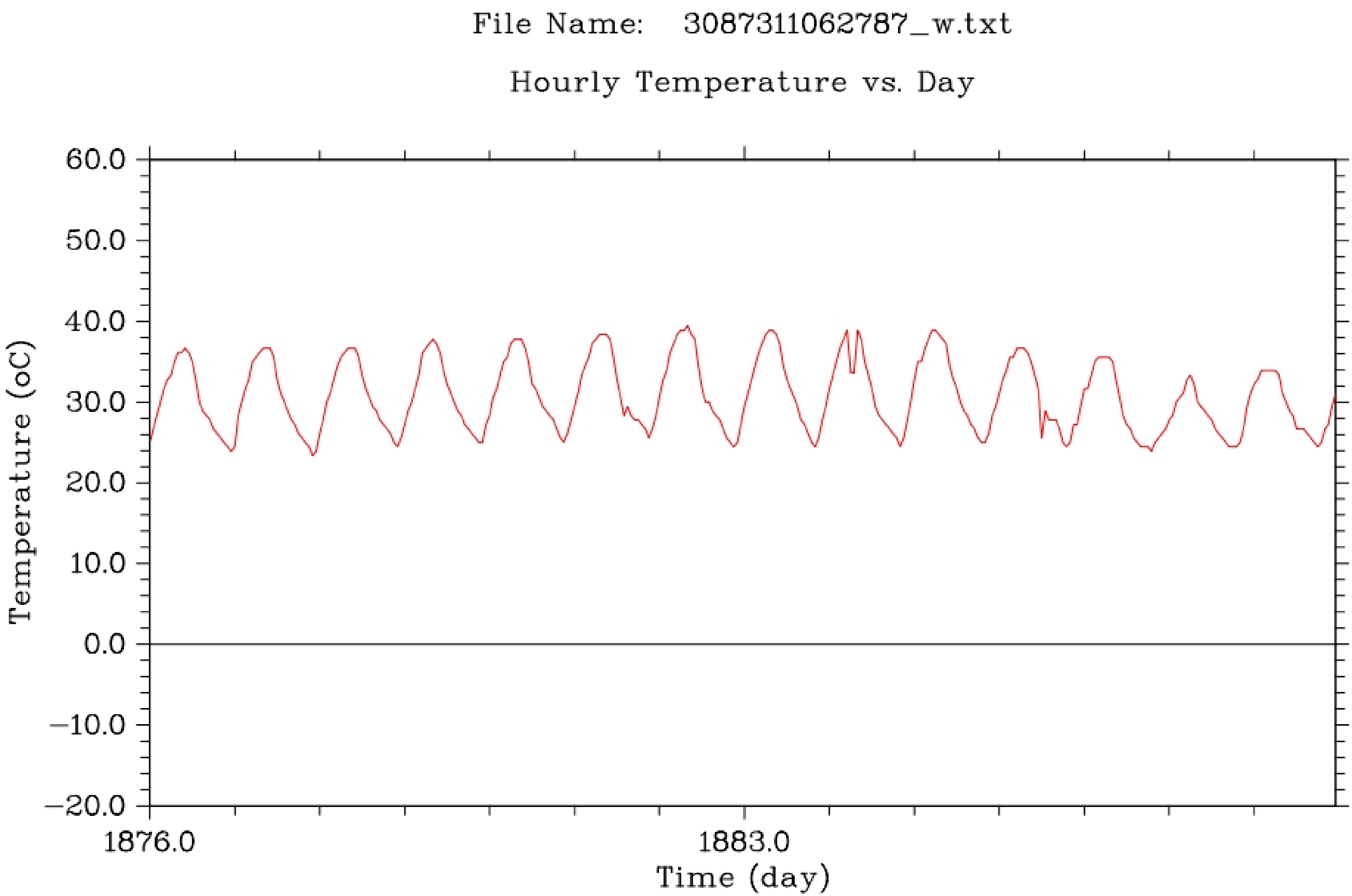
Wavelet Analysis for Temperature Data



Wavelet Analysis for Temperature Data

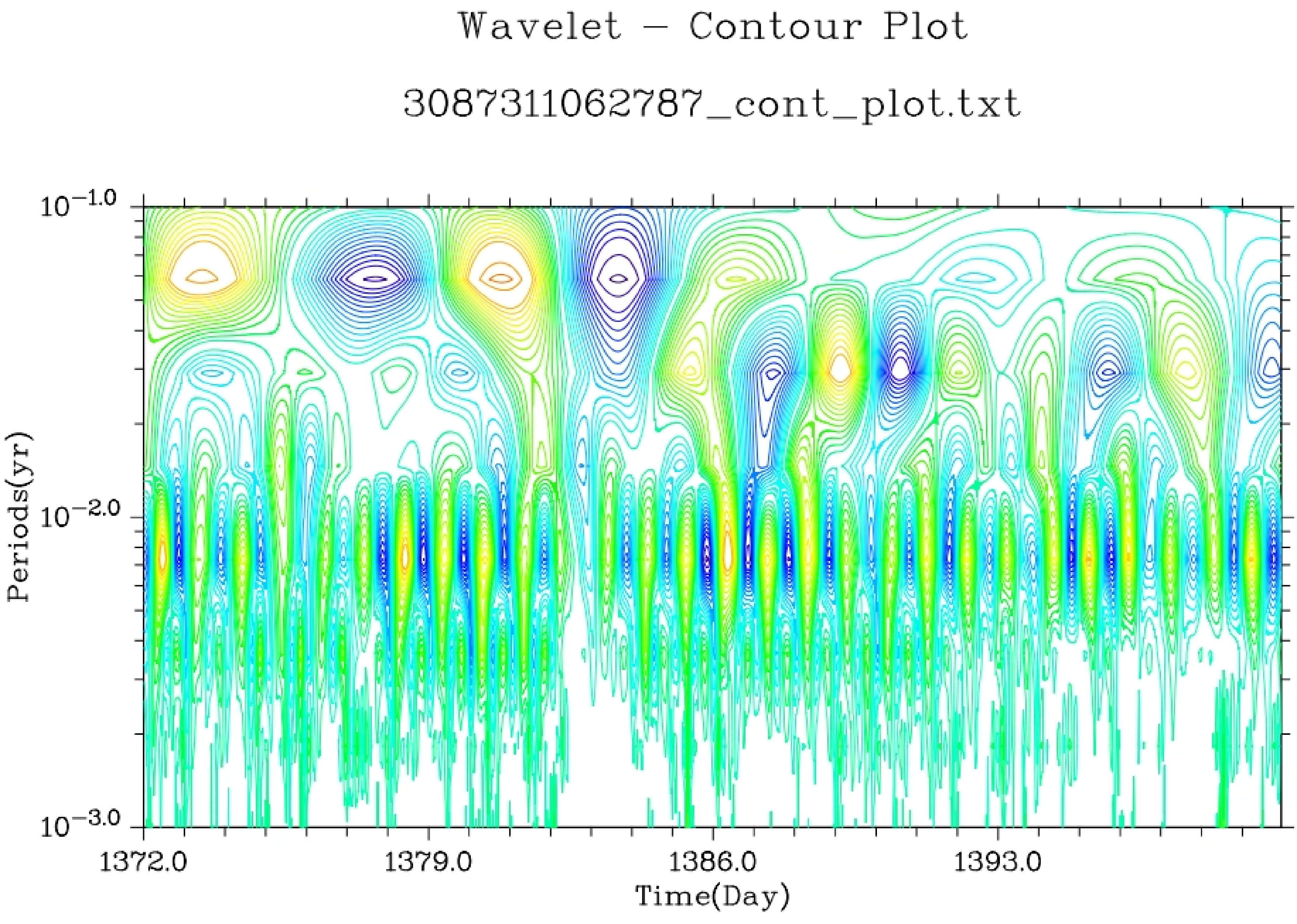


Wavelet Analysis for Temperature Data

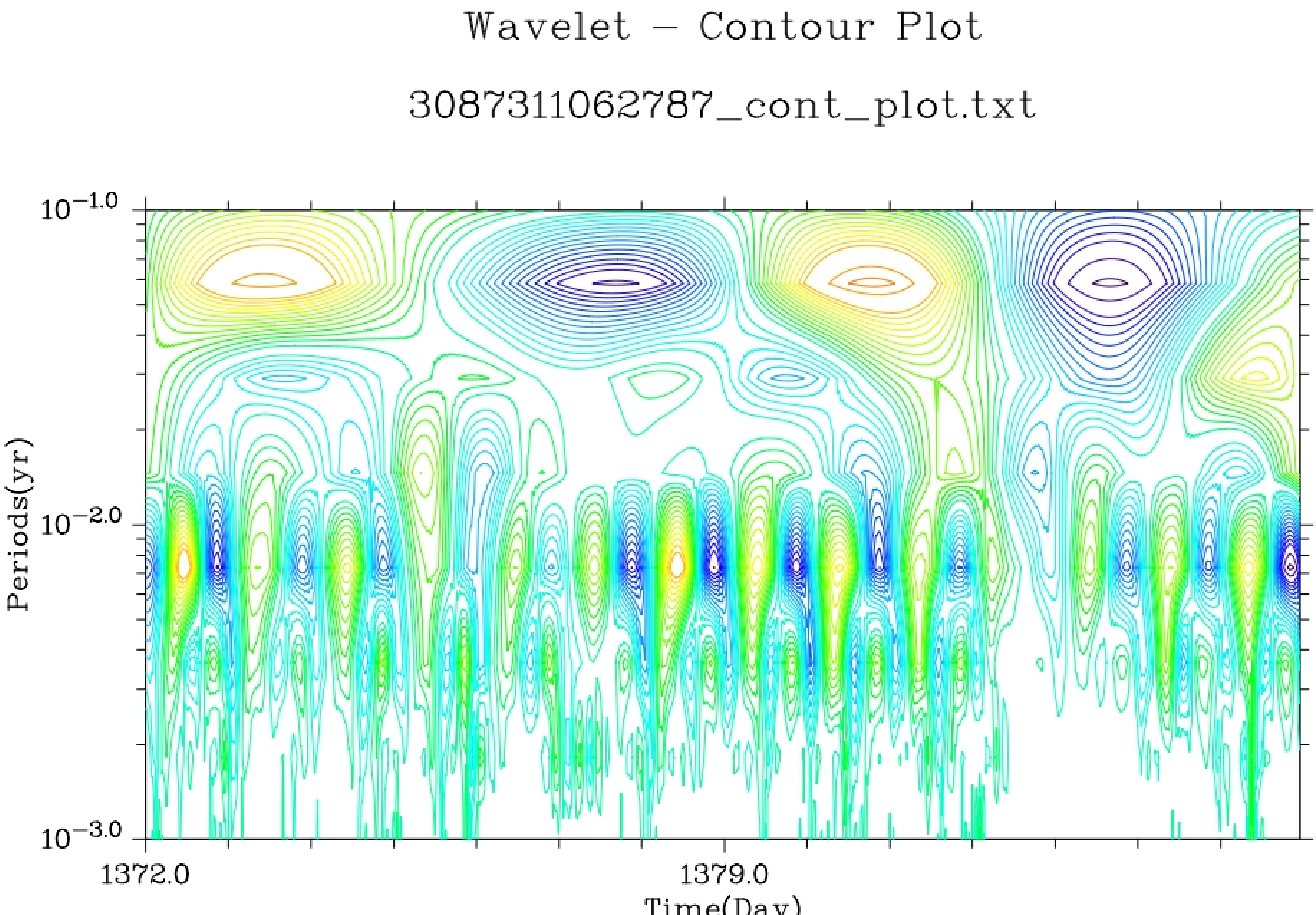


Original Signal

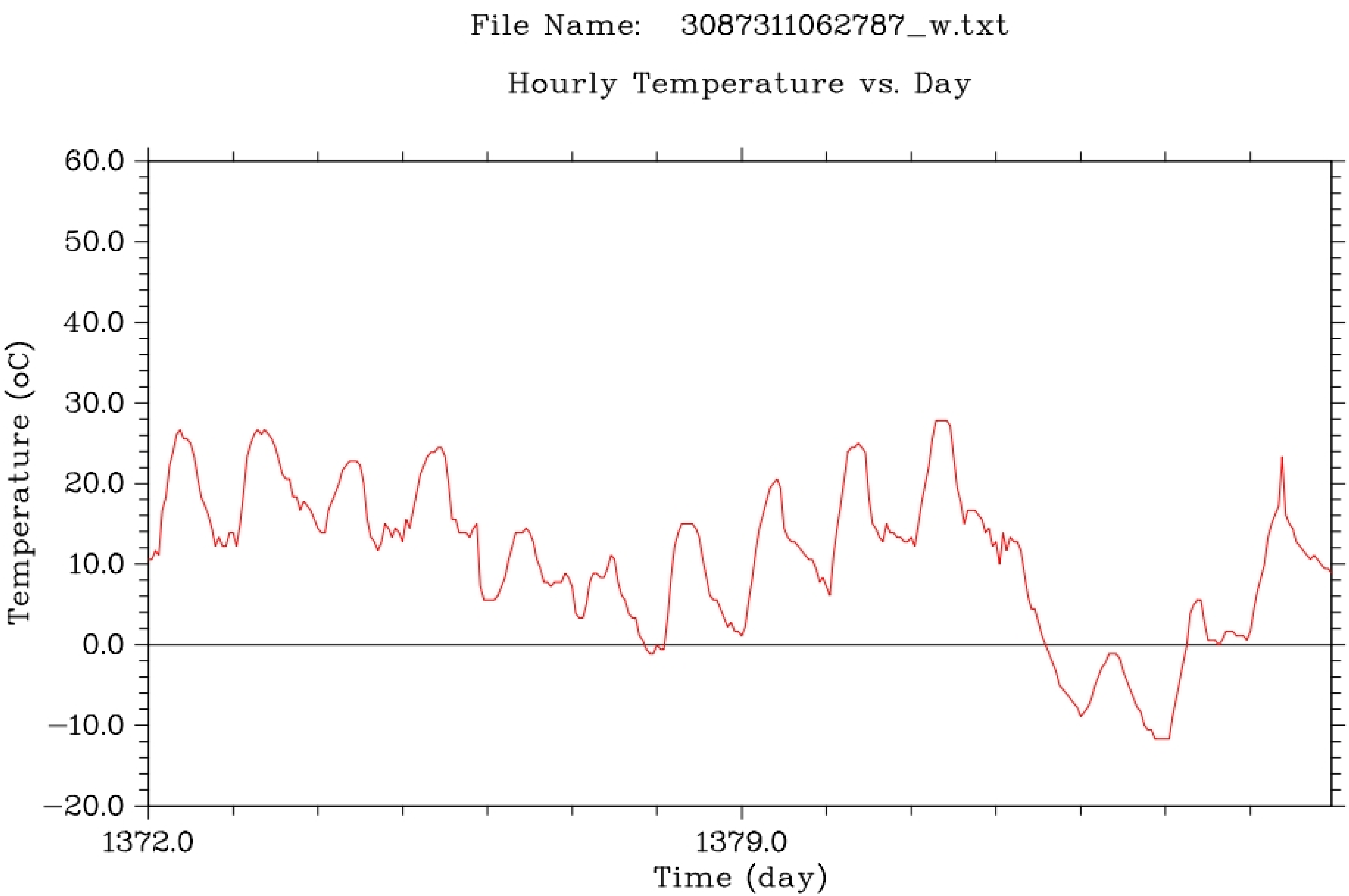
Wavelet Analysis for Temperature Data



Wavelet Analysis for Temperature Data



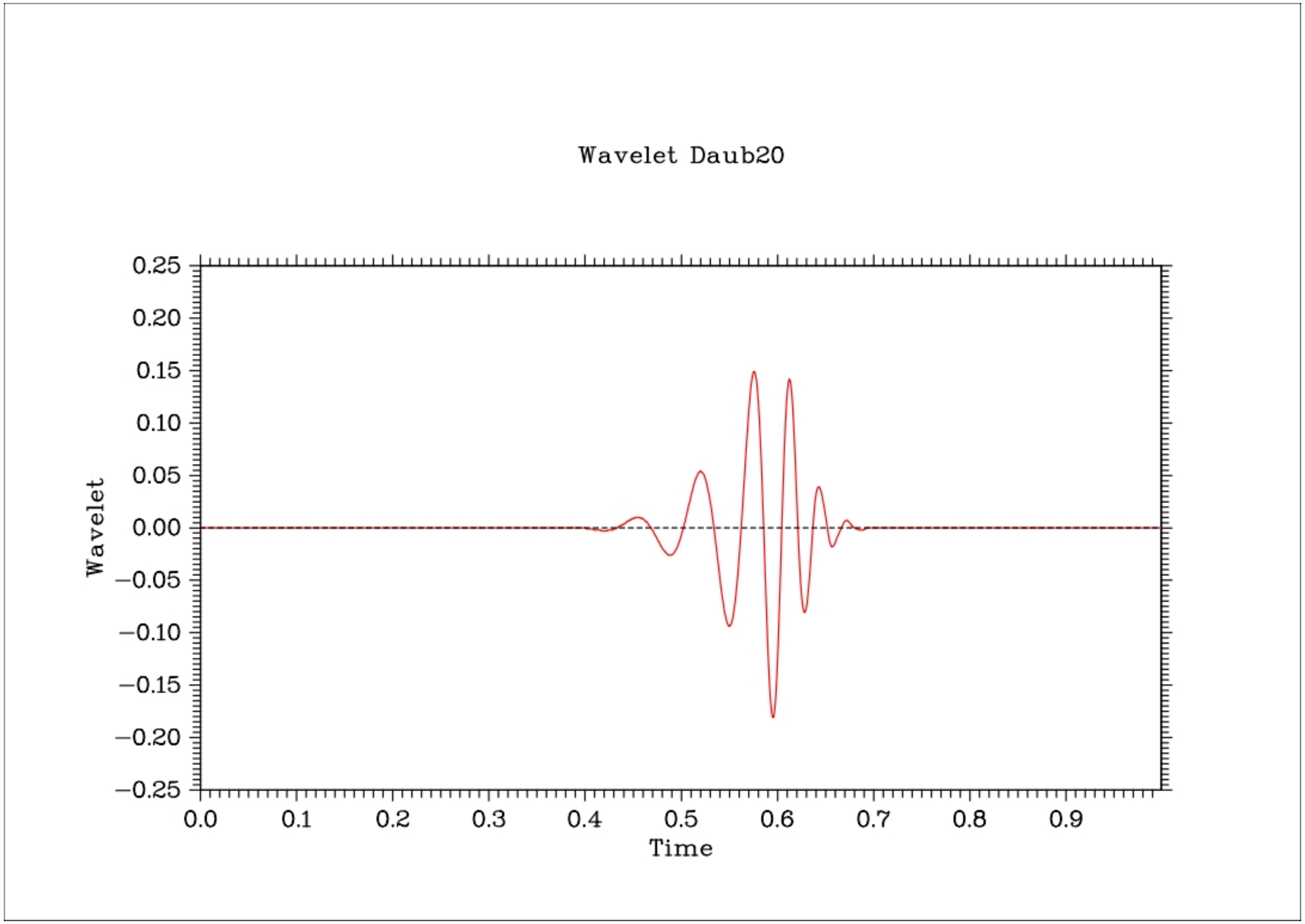
Wavelet Analysis for Temperature Data



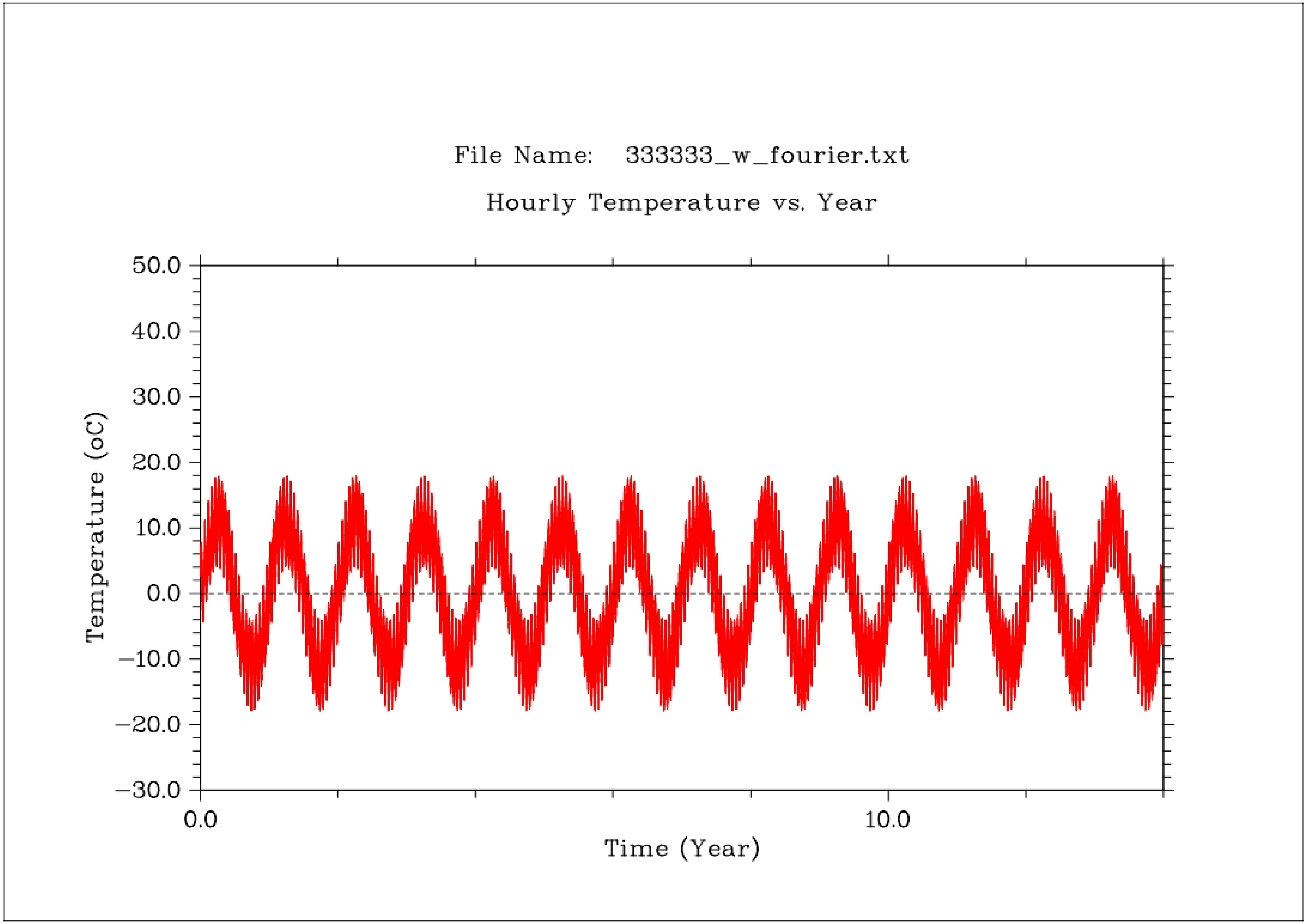
Original Signal

Thank You.

Any Questions?



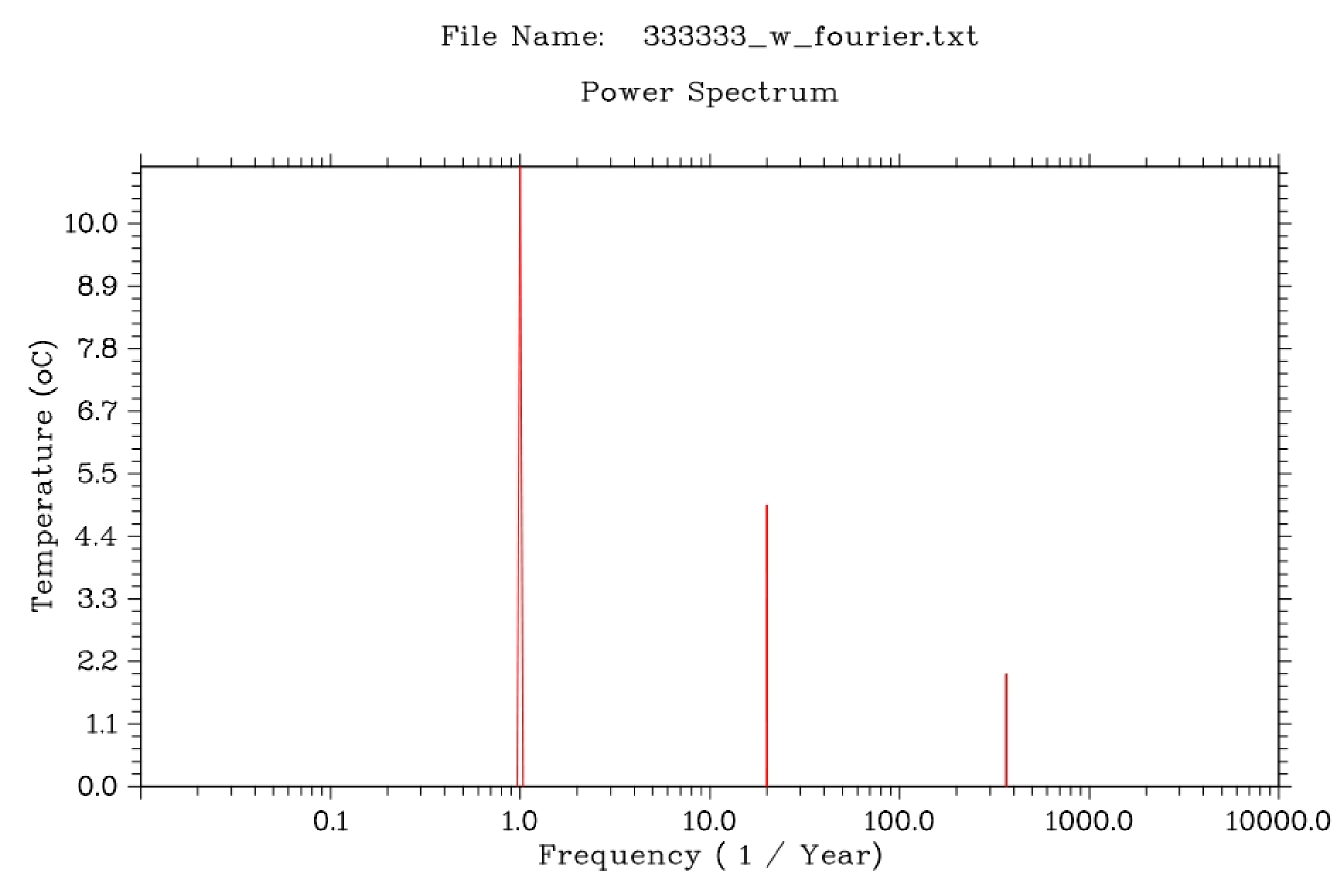
Three Frequency Test



$$y(t) = 11.0 \sin(2\pi\, 1.0\, s^{-1}\, t) + 2.0 \sin(2\pi\, 365.25\, s^{-1}\, t) + 5.0 \sin(2\pi\, 20\, s^{-1}\, t)$$

Three Frequency Test – Power Spectrum

$$y(t) = 11.0 \sin(2\pi\, 1.0\, s^{-1} t) + 2.0 \sin(2\pi\, 365.25\, s^{-1} t) + 5.0 \sin(2\pi\, 20\, s^{-1} t)$$



i= 28	f= 1.000000	a= 10.99	b= 0.000000	A= 10.99
i= 560	f= 20.000000	a= 4.99	b= 0.000000	A= 4.99
i= 10227	f= 365.250000	a= 1.99	b= 0.000000	A= 1.99

Using a packaged DFT to do Fourier Analysis

Temperature – Power Spectrum

```
#include <sunperf.h>
.
.
.
  ezffti(n, wsave);
  ezfftf(n, r, azero, a, b, wsave);
.
.
.

cc test_fft_perf_file.c -lm  -v -xlic_lib=sunperf -o test_fft_perf_file
```

Temperature – Power Spectrum

n (input)

Number of Data Points.

r (input)

A real array of length n containing the sequence to be transformed. On exit, r is unchanged.

azero (output)

On exit, the sum from i=1 to i=n of r(i)/n, i.e. The average of the signal.

a (input/output)

On entry, arrays that contain the remaining Fourier coefficients. The amplitude of the cosine waves.

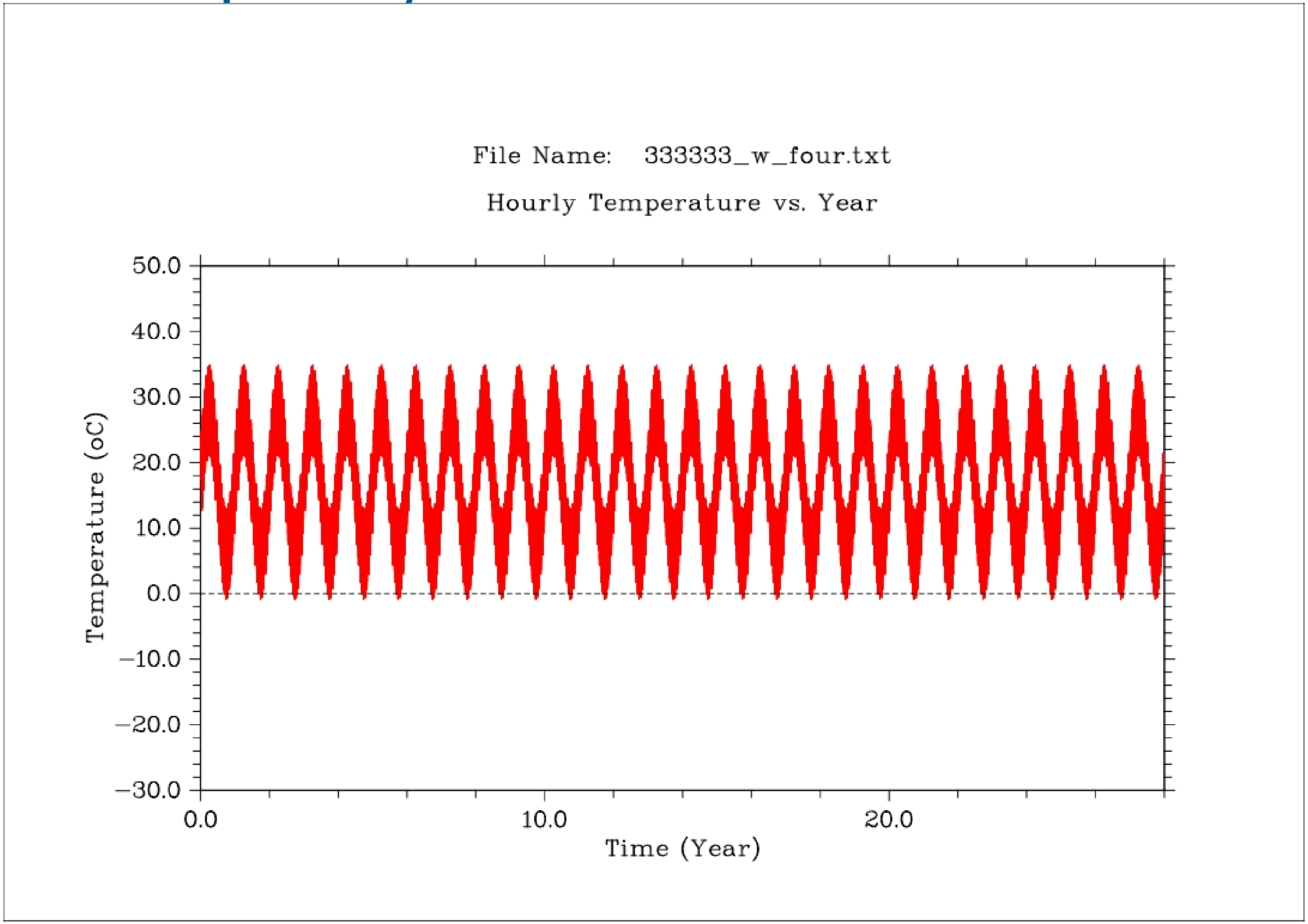
b (input/output)

On entry, arrays that contain the remaining Fourier coefficients. The Amplitude of the sine waves.

wsave (input)

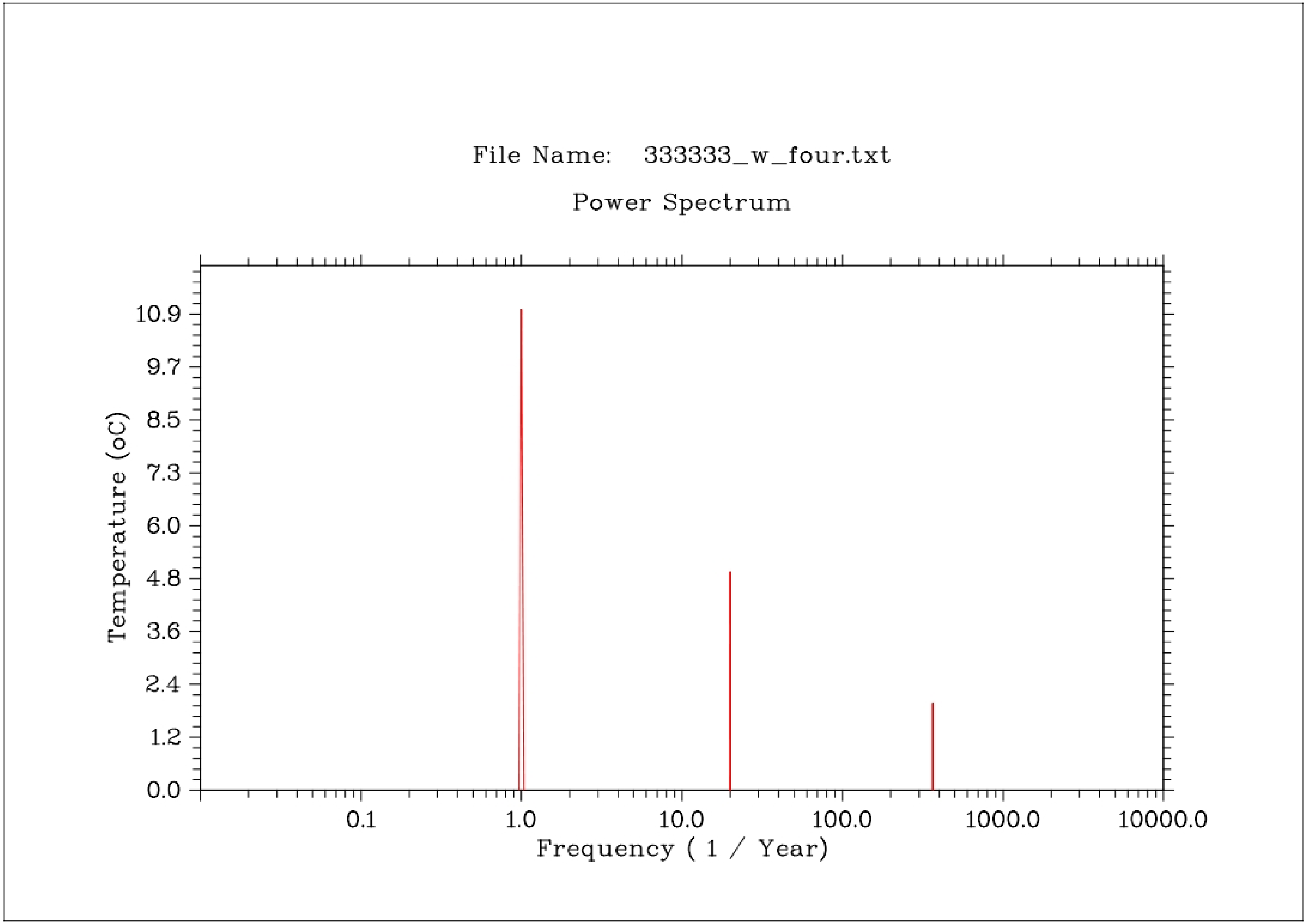
On entry, an array with dimension of at least (3 *N + 15), initialized by EZFFTI. A working array.

Three Frequency Test



$$y(t) = 11.0\sin(2\pi\, 1.0\, s^{-1} t) + 2.0\sin(2\pi\, 365.25\, s^{-1} t) + 5.0\sin(2\pi\, 20.0\, s^{-1} t) + 17$$

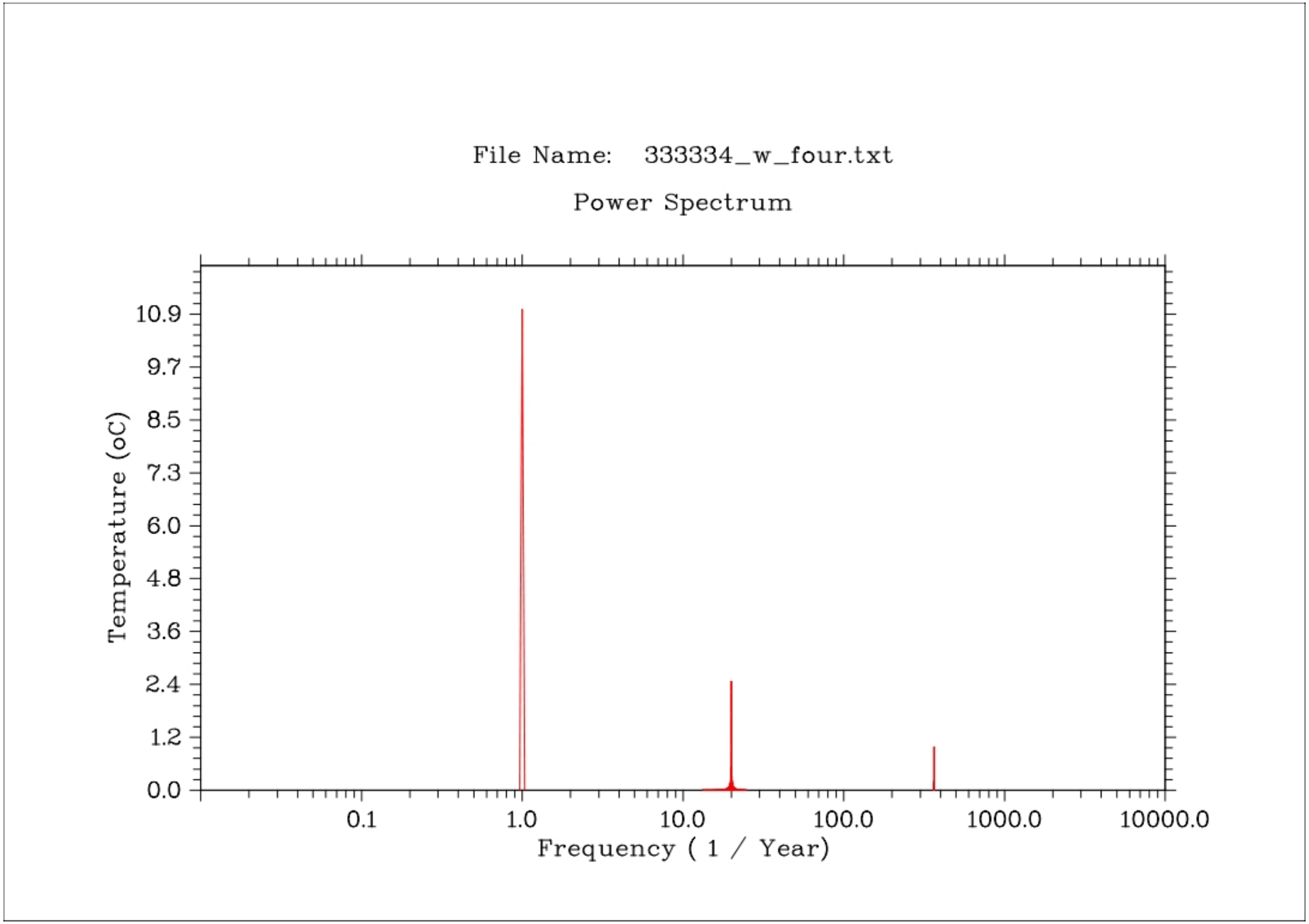
Three Frequency Test – Power Spectrum



azero: 16.999989

i= 28	f= 1.000000	a= 0.000003	b= 11.000760	A= 11.000760
i= 560	f= 20.000002	a= -0.000012	b= 5.000956	A= 5.000956
i= 10227	f= 365.250000	a= -0.000082	b= 2.000099	A= 2.000099

Three Frequency Test – Power Spectrum



azero: 16.9999

i= 28 f= 1.000000 a= -0.000121 b= 11.000760 A= 11.000760

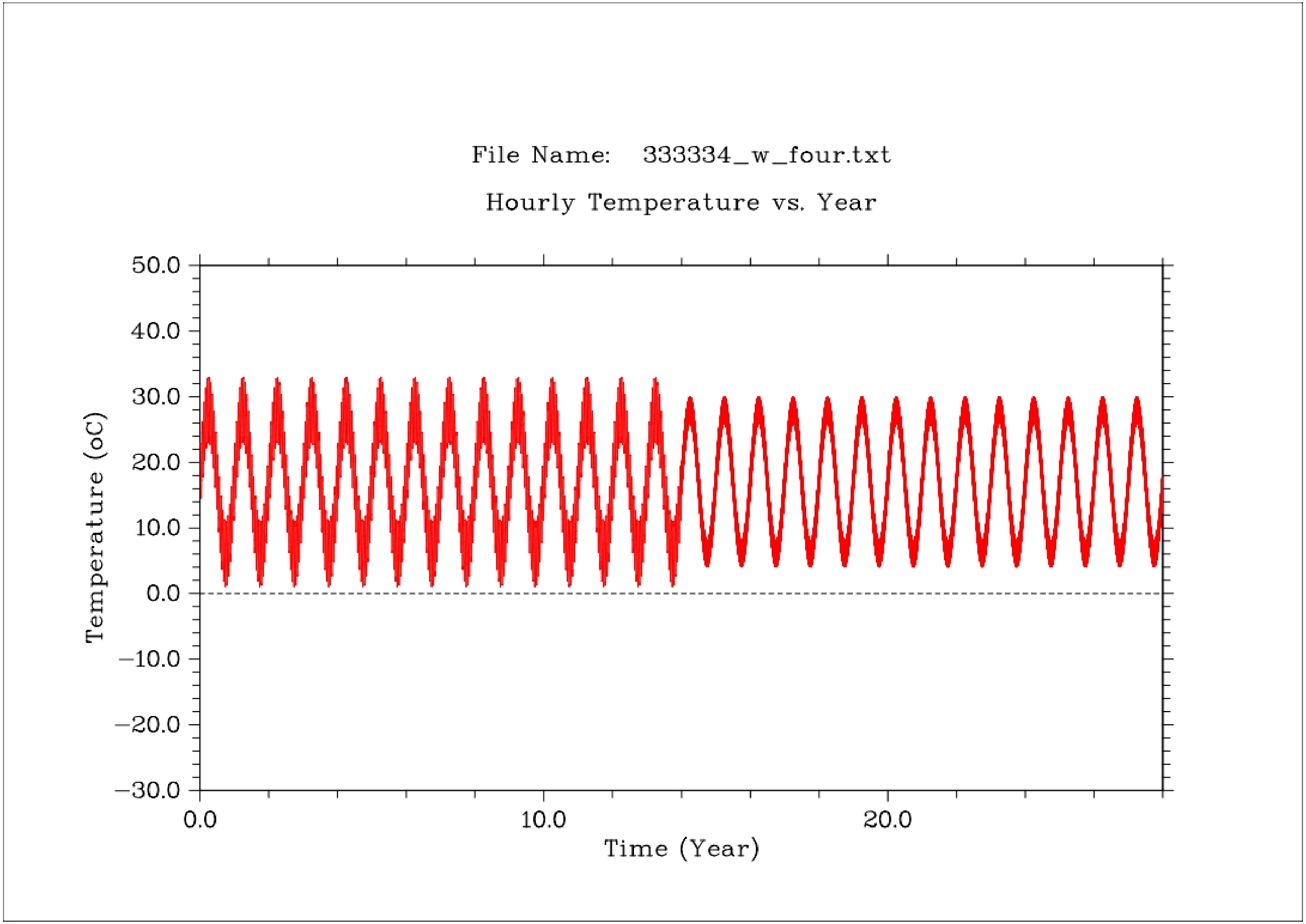
i= 559 f= 19.964287 a= 1.593312 b= -0.000105 A= 1.593312

i= 560 f= 20.000002 a= -0.000225 b= 2.500476 A= 2.500476

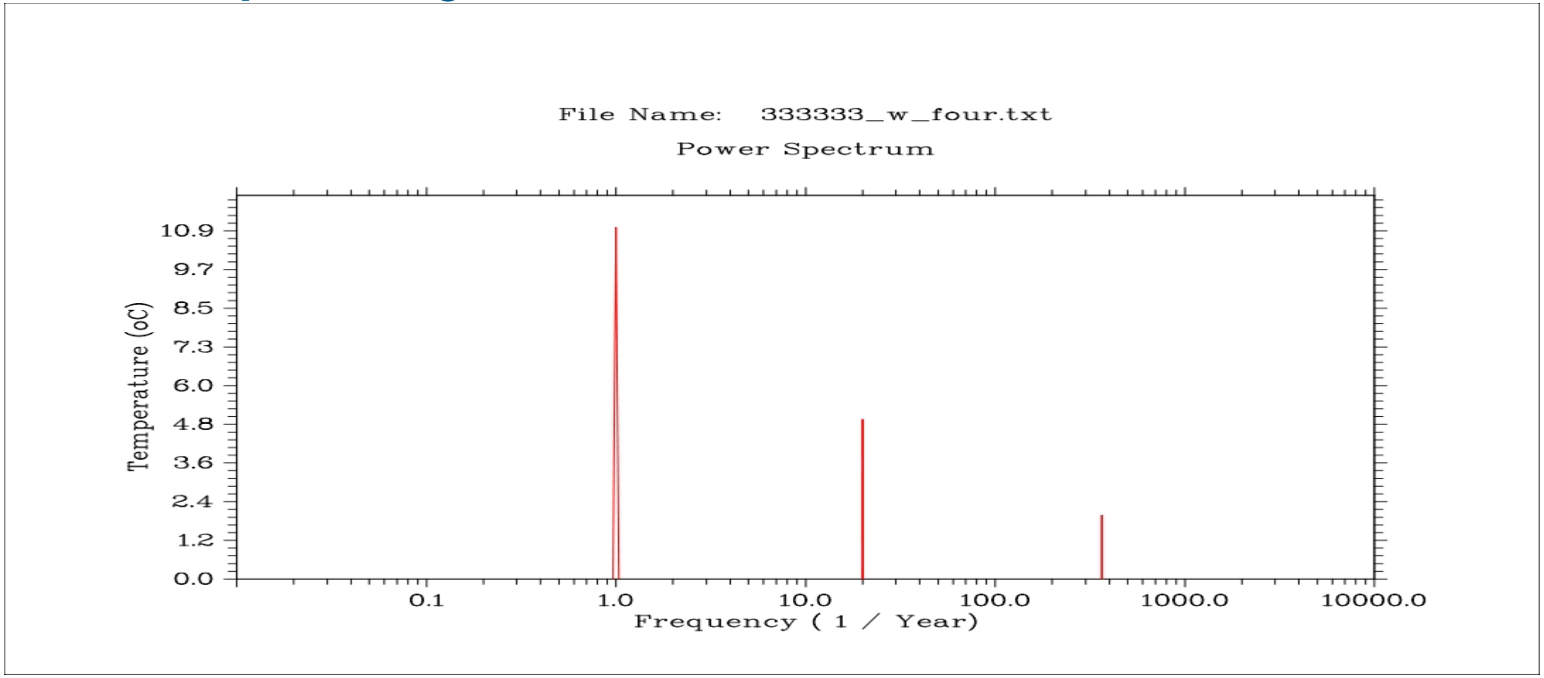
i= 561 f= 20.035715 a= -1.590393 b= -0.000237 A= 1.590393

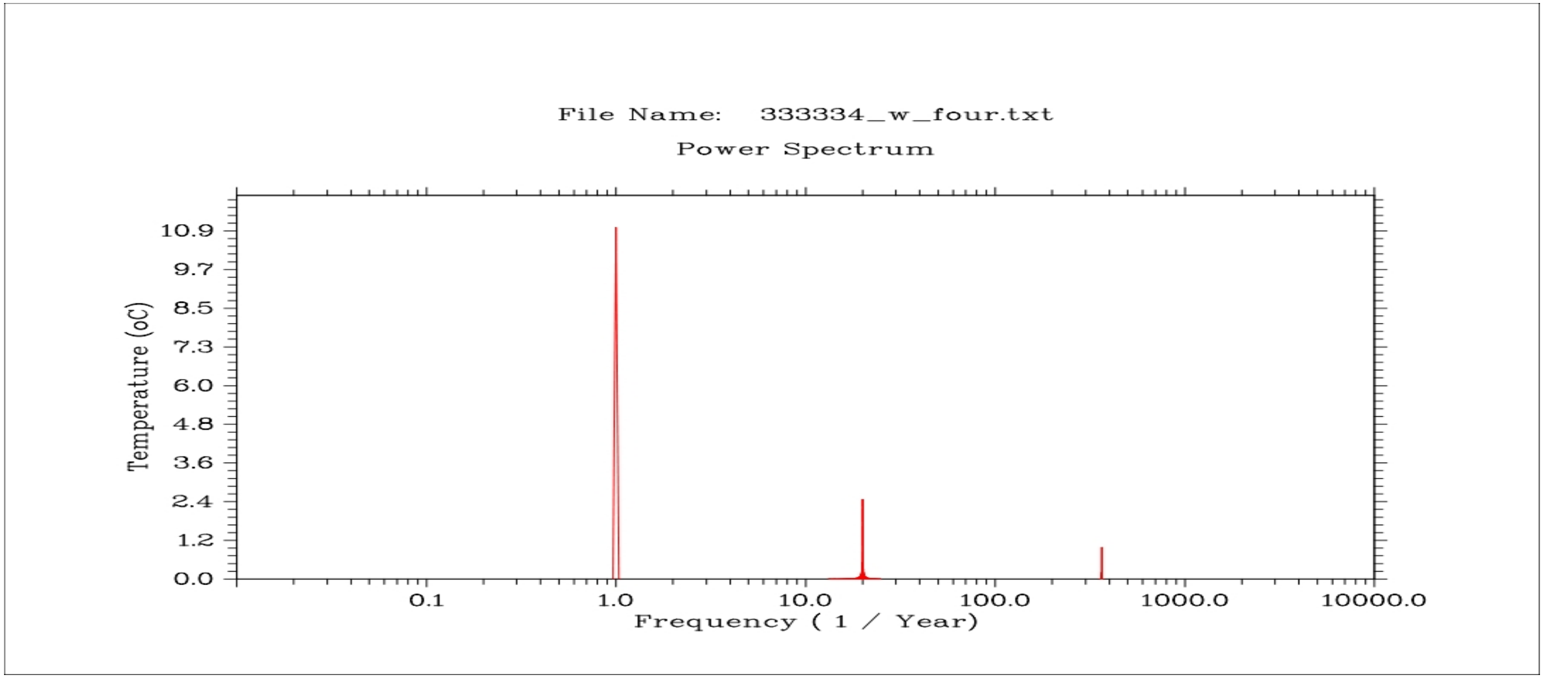
i= 10227 f= 365.250000 a= -0.000078 b= 1.000049 A= 1.000049

Three Frequency Test – Power Spectrum

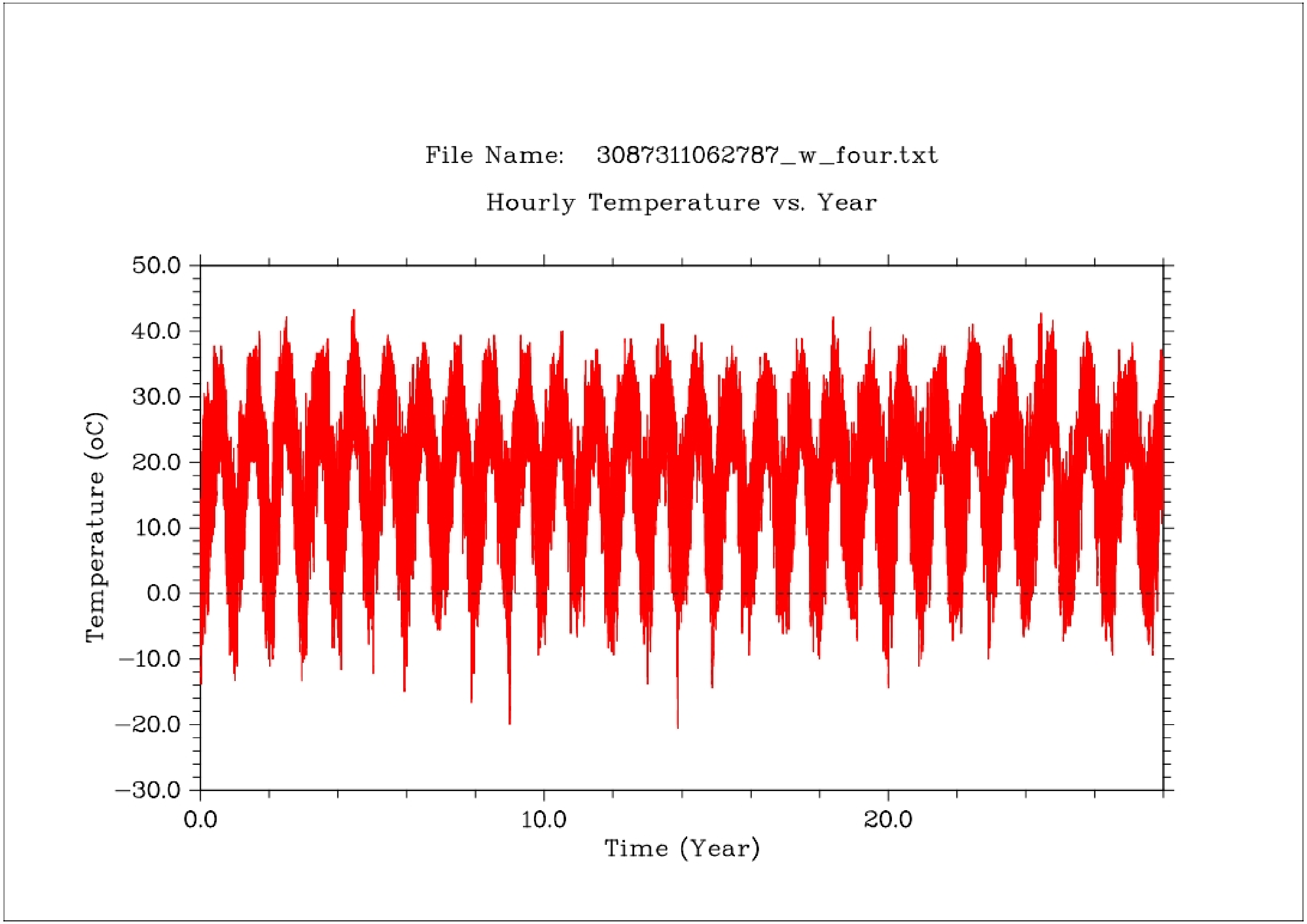


Three Frequency Test – Problem With FFT



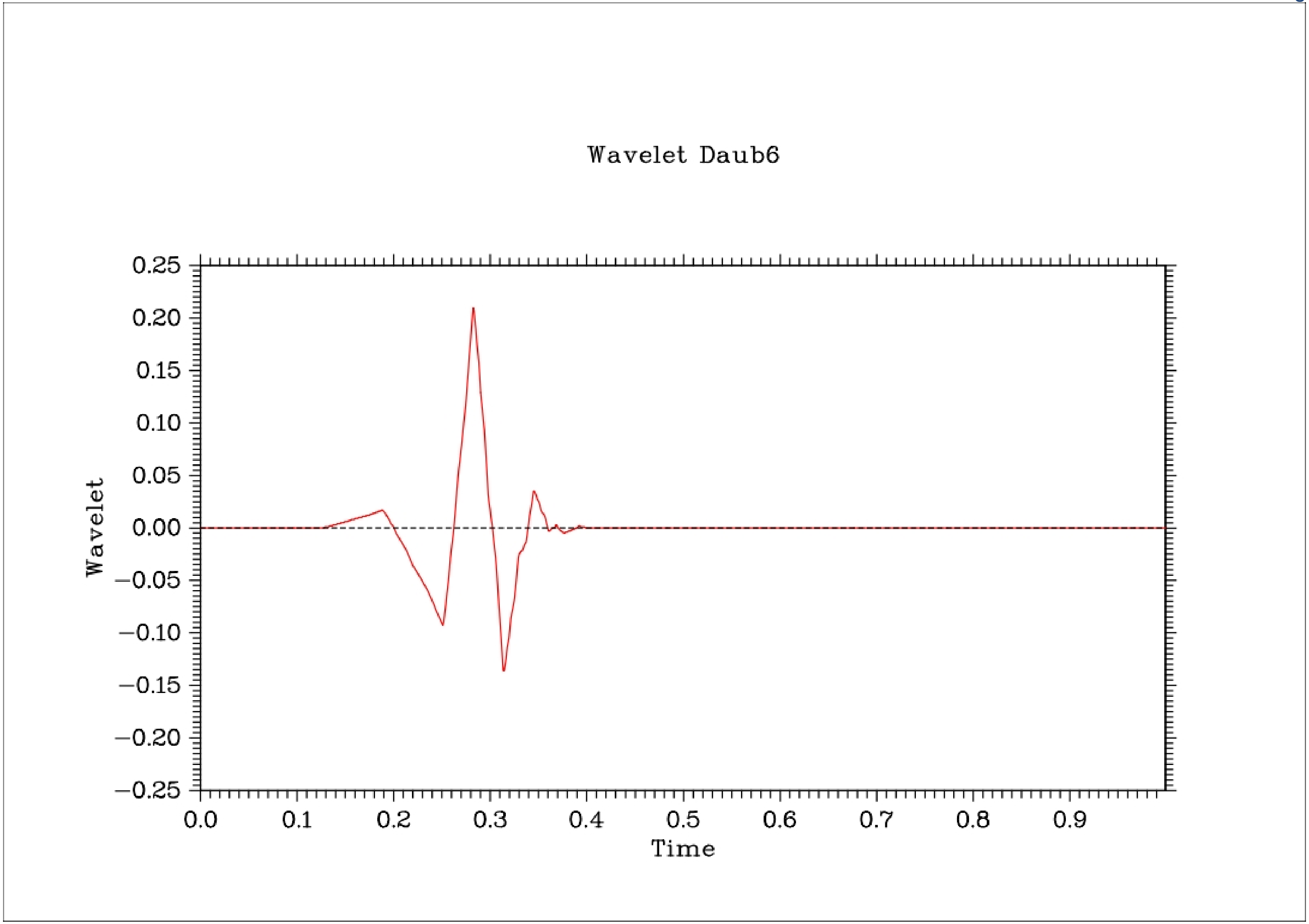


Temperature – Power Spectrum

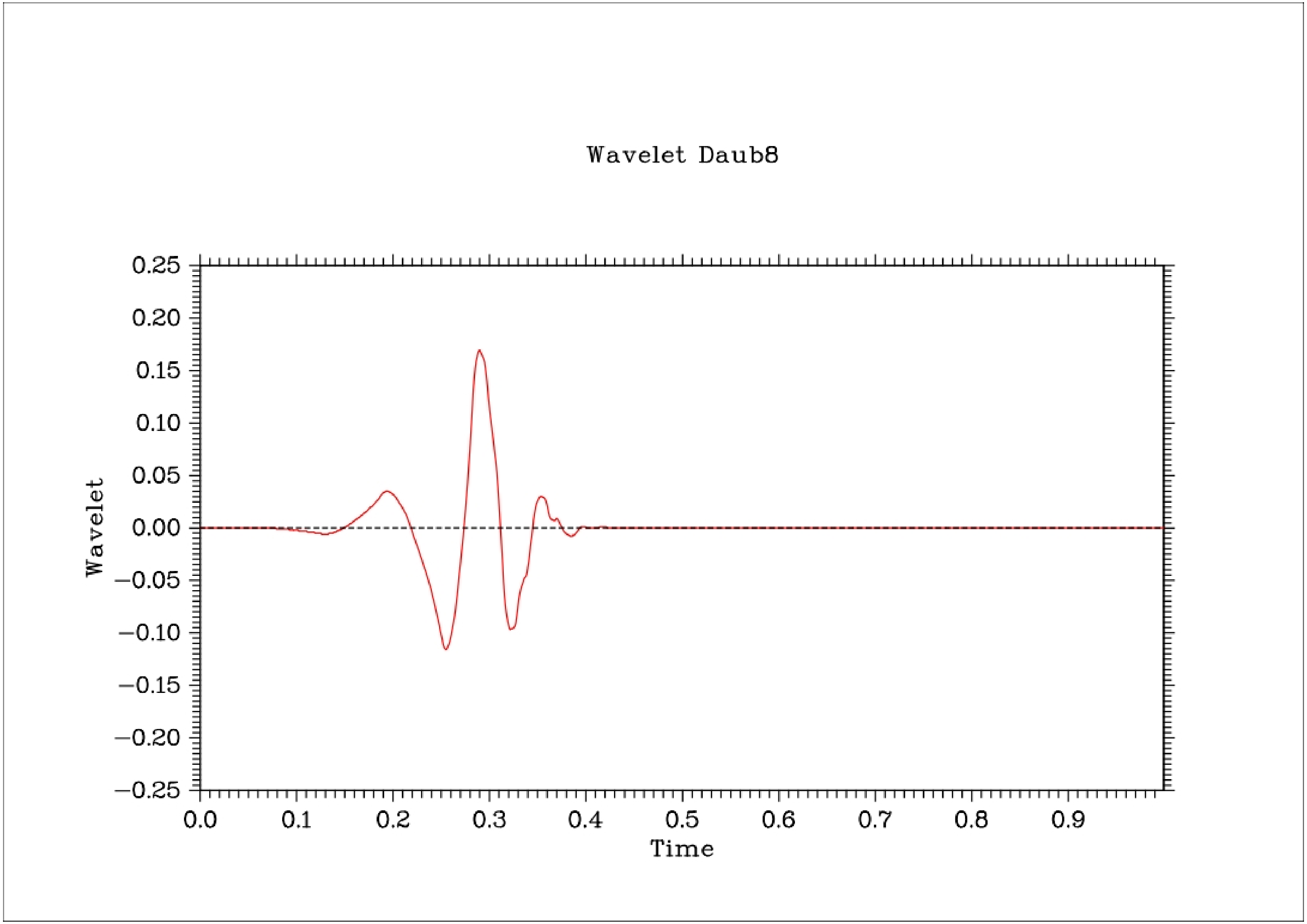


Sample of Wavelets : Daubechies

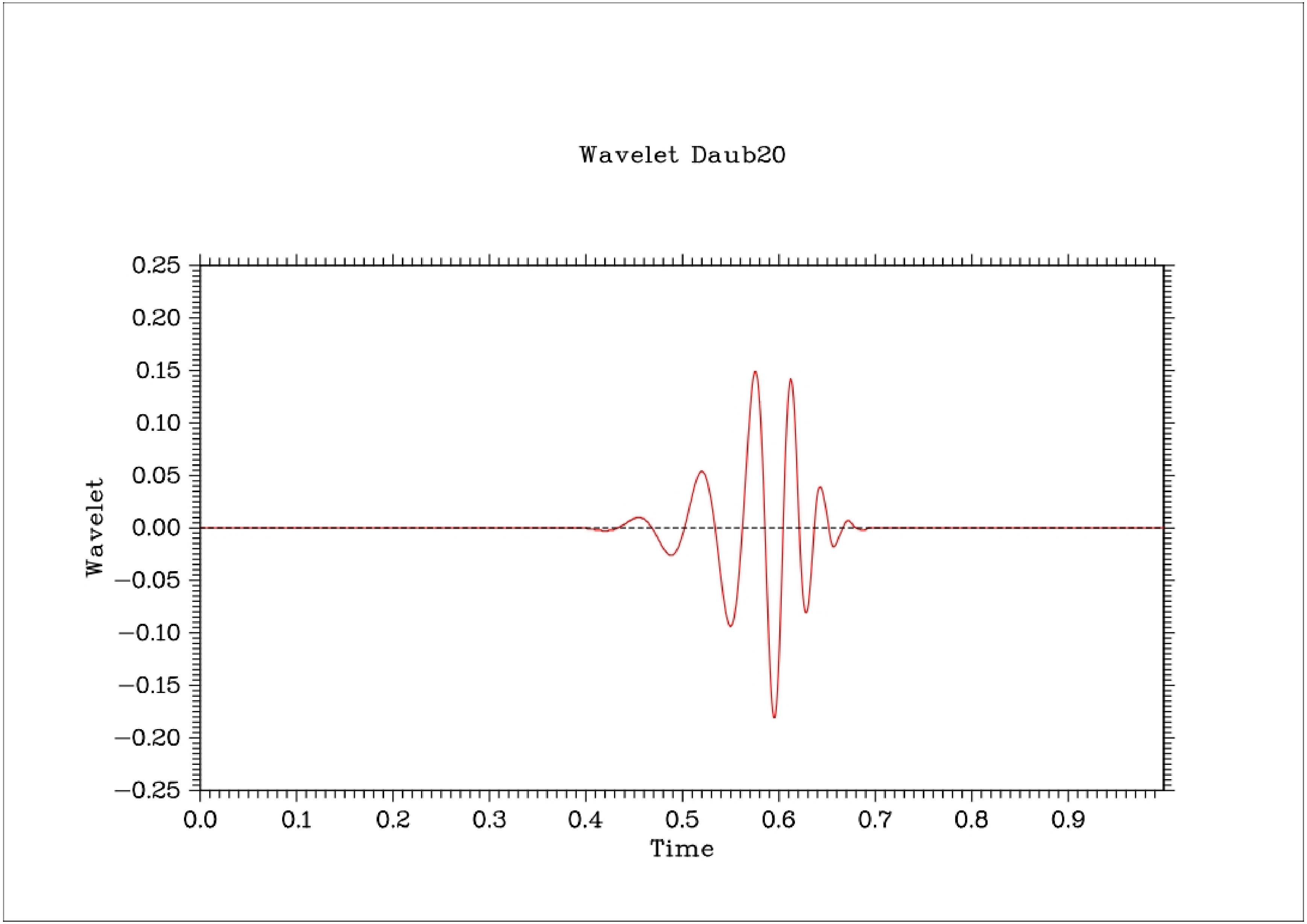
Sample of Wavelets : Daubechies



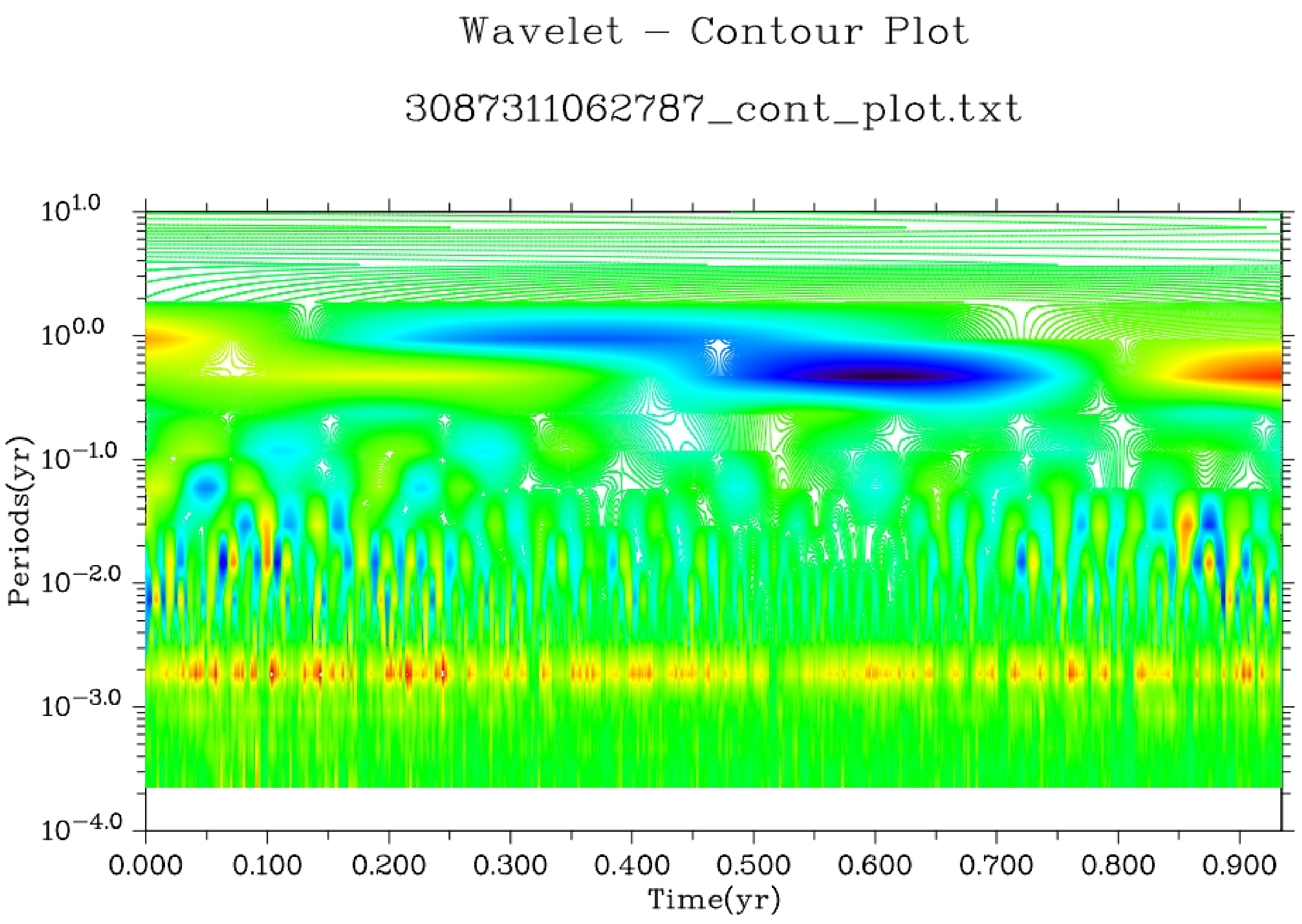
Sample of Wavelets : Daubechies



Sample of Wavelets : Daubechies



Sample of Wavelets : Daubechies



Wavelet – Contour Plot
3087311062787_cont_plot.txt
Periods(yr)
10^1.0
10^0.0
10^−1.0
10^−2.0
10^−3.0
10^−4.0
0.000
0.100
0.200
0.300
0.400
0.500
0.600
0.700
0.800
0.900
Time(yr)

Discrete Wavelet Transformation (DWT)

$$
\begin{bmatrix}
c_0 & c_1 & c_2 & c_3 & & & & & & & \\
c_3 & -c_2 & c_1 & -c_0 & & & & & & & \\
 & & c_0 & c_1 & c_2 & c_3 & & & & & \\
 & & c_3 & -c_2 & c_1 & -c_0 & & & & & \\
\ldots. & & & & . & & & & & & \\
\ldots. & & & & & . & & & & & \\
\ldots. & & & & & & . & & & & \\
 & & & & & & & c_0 & c_1 & c_2 & c_3 \\
 & & & & & & & c_3 & -c_2 & c_1 & -c_0 \\
c_2 & c_3 & & & & & & & & c_0 & c_1 \\
c_1 & -c_0 & & & & & & & & c_3 & -c_2
\end{bmatrix}
$$

Transformation matrix of Daub4.

More Slides if Needed….

Historical Perspective – Heat Equation:

$x=0$ $\qquad\qquad\qquad\qquad$ $x=\pi$

$$u_t(x,t)=u_{xx}(x,t) \quad t>0 \;,\; 0\le x\le \pi$$

$$u(x,0)=f(x) \quad 0\le x\le \pi$$

$$u(0,t)=A$$

$$u(\pi,t)=B$$

$u(x,t)$ •Solution to differential equations represent temperature of a rod of length π at position x and time t with initial temperature (at t =0) given by f(x), where temperature at the ends of the rod, x=0 and x=p, are kept at A and B.

•Consider special case where A=B=0.

Assume solution has the form:

$$u(x,t)=X(x)T(t)$$

Use separation of variables.

Historical Perspective – Heat Equation:

$x=0 \qquad x=\pi$

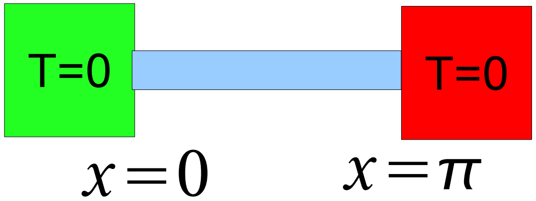
$$u_t(x,t)=u_{xx}(x,t) \qquad t>0\;,\;0\le x\le\pi$$
$$u(x,0)=f(x) \qquad 0\le x\le\pi$$
$$u(0,t)=A$$
$$u(\pi,t)=B$$

$$u(x,t)=X(x)T(t)$$ Where $T(t)\;\; t\ge 0$, $X(x)\;\; 0\le x\le\pi$

$$u_t=u_{xx}$$
$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial^2 x}$$

$$\frac{\partial u}{\partial t}=T'(t)$$
$$\frac{\partial^2 u}{\partial^2 x}=X''(x)$$

Historical Perspective – Heat Equation:



$$u_t(x,t)=u_{xx}(x,t) \qquad t>0 \;,\; 0\le x\le\pi$$
$$u(x,0)=f(x) \qquad 0\le x\le\pi$$
$$u(0,t)=A$$
$$u(\pi,t)=B$$

$$u(x,t)=X(x)T(t)$$ Where $T(t)\;\; t\ge 0$, $X(x)\;\; 0\le x\le\pi$

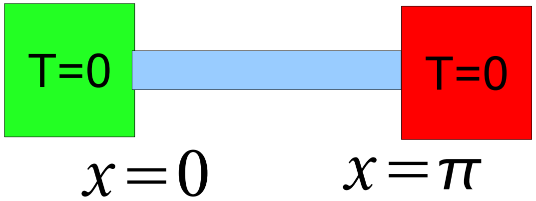
$$u_t=u_{xx}$$
$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial^2 x}$$
$$X(x)T'(t)=X''(x)T(t)$$
$$\frac{T'(t)}{T(t)}=\frac{X''(x)}{X(x)}$$

$$\frac{\partial u}{\partial t}=X(x)T'(t)$$
$$\frac{\partial^2 u}{\partial^2 x}=T(t)X''(x)$$

Since left hand side of equation depends only on t and right hand side depends only on x, then:

$$\frac{T'(t)}{T(t)}=c \qquad \frac{X''(x)}{X(x)}=c$$

Historical Perspective – Heat Equation:



$$u_t(x,t)=u_{xx}(x,t) \qquad t>0\ ,\ 0\le x\le \pi$$
$$u(x,0)=f(x) \qquad 0\le x\le \pi$$
$$u(0,t)=A$$
$$u(\pi,t)=B$$

$$\frac{T'(t)}{T(t)}=c \quad \frac{X''(x)}{X(x)}=c$$

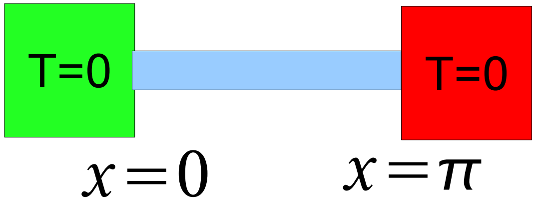
$$T'(t)=c\,T(t)$$
$$T'(t)=Ce^{ct}\ \textit{for some constant}\ C$$

$c<0$ or else $|T(t)|$ and hence $|u(x,t)|$, would increase as $t\to\infty$, let $c=-\lambda^2<0$

$$T'(t)=Ce^{ct}=C\,e^{-\lambda^2 t}$$

$$X''(x)+\lambda^2 X(x)=0 \quad 0\le x\le \pi \quad X(0)=X(\pi)=0$$

Historical Perspective – Heat Equation:



$$u_t(x,t)=u_{xx}(x,t) \qquad t>0\ ,\ 0\le x\le\pi$$
$$u(x,0)=f(x) \qquad 0\le x\le\pi$$
$$u(0,t)=A$$
$$u(\pi,t)=B$$

$$\frac{T'(t)}{T(t)}=c \quad \frac{X''(x)}{X(x)}=c$$

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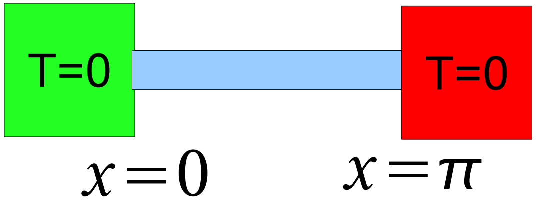
$c<0$ or else $|T(t)|$ and hence $|u(x,t)|$, would increase as $t\to\infty$, let $c=-\lambda^2<0$

$$T'(t)=Ce^{ct}=C\,e^{-\lambda^2 t}$$

$$X''(x)+\lambda^2X(x)=0 \quad 0\le x\le\pi \quad X(0)=X(\pi)=0$$
$$X(x)=a\cos(\lambda x)+b\sin(\lambda x)$$

Historical Perspective – Heat Equation:



$$u_t(x,t)=u_{xx}(x,t) \qquad t>0\ ,\ 0\le x\le \pi$$
$$u(x,0)=f(x) \qquad 0\le x\le \pi$$
$$u(0,t)=A$$
$$u(\pi,t)=B$$

$$\frac{T'(t)}{T(t)}=c \quad \frac{X''(x)}{X(x)}=c$$

Using Boundary Conditions:

$$T'(t)=Ce^{ct}=C\,e^{-\lambda^2 t}$$

$$X''(x)+\lambda^2 X(x)=0 \quad 0\le x\le \pi \quad X(0)=X(\pi)=0$$
$$X(x)=a\cos(\lambda x)+b\sin(\lambda x)$$

$X(0)=0$ therefore $a=0$

$X(0)=X(\pi)=0$ therefore b must be an integer. Let b = k so:

$$u_k(x,t)=X_k(x)\,T_k(t)=b_k e^{-k^2 t}\sin(kx)$$

Historical Perspective – Heat Equation:

$x=0$ $\quad$ $x=\pi$

$$u_t(x,t)=u_{xx}(x,t) \qquad t>0 \;,\; 0\le x\le \pi$$
$$u(x,0)=f(x) \qquad 0\le x\le \pi$$
$$u(0,t)=A$$
$$u(\pi,t)=B$$

$$\frac{T'(t)}{T(t)}=c \quad \frac{X''(x)}{X(x)}=c$$

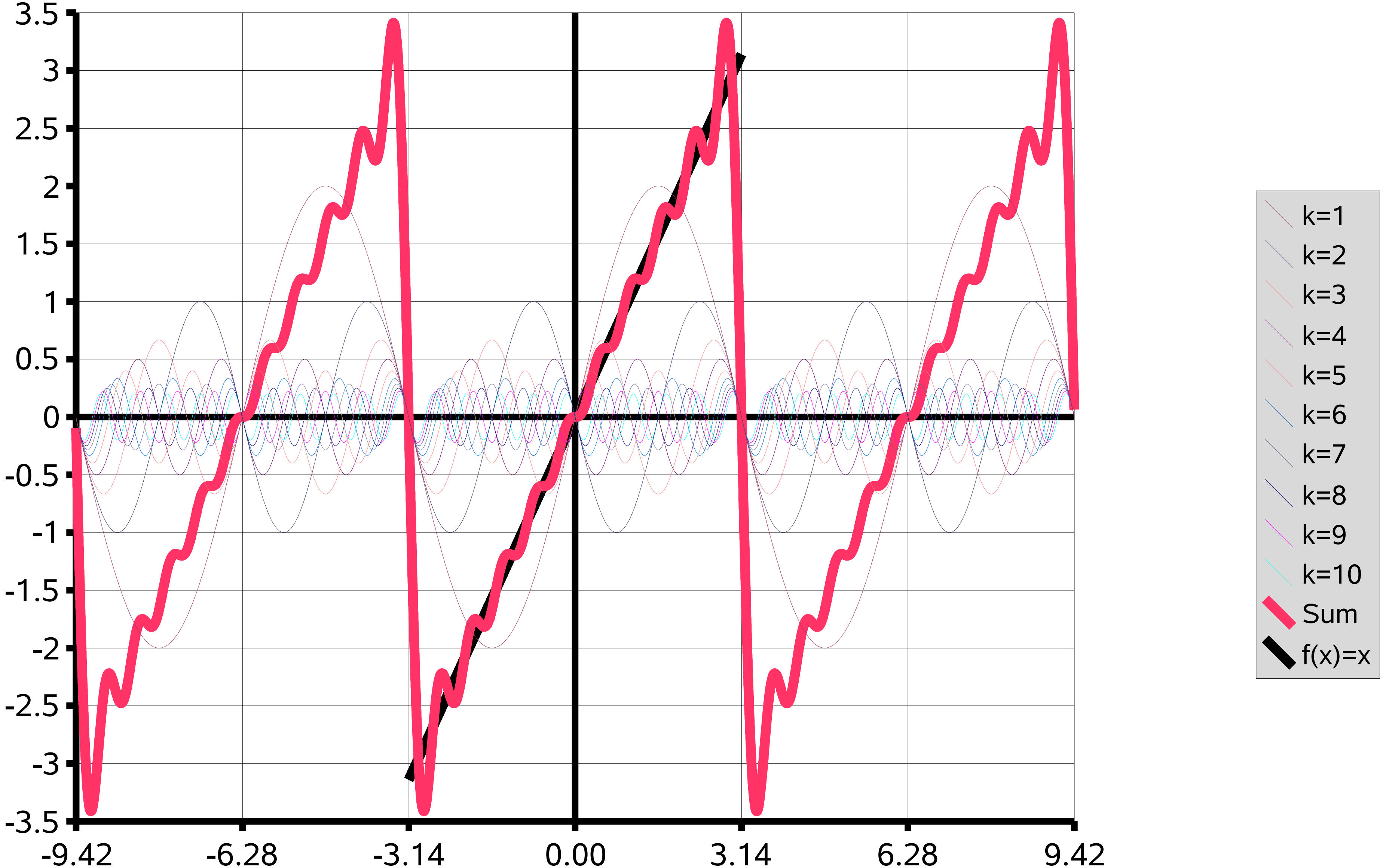
Using Initial Conditions: $u(x,0)=f(x)$

$$u(x,t)=\sum_{k=1}^{\infty} u_k(x,t)=\sum_{k=1}^{\infty} b_k e^{-k^2 t}\sin(kx)$$

but $u(x,t=0)=f(x)$ so

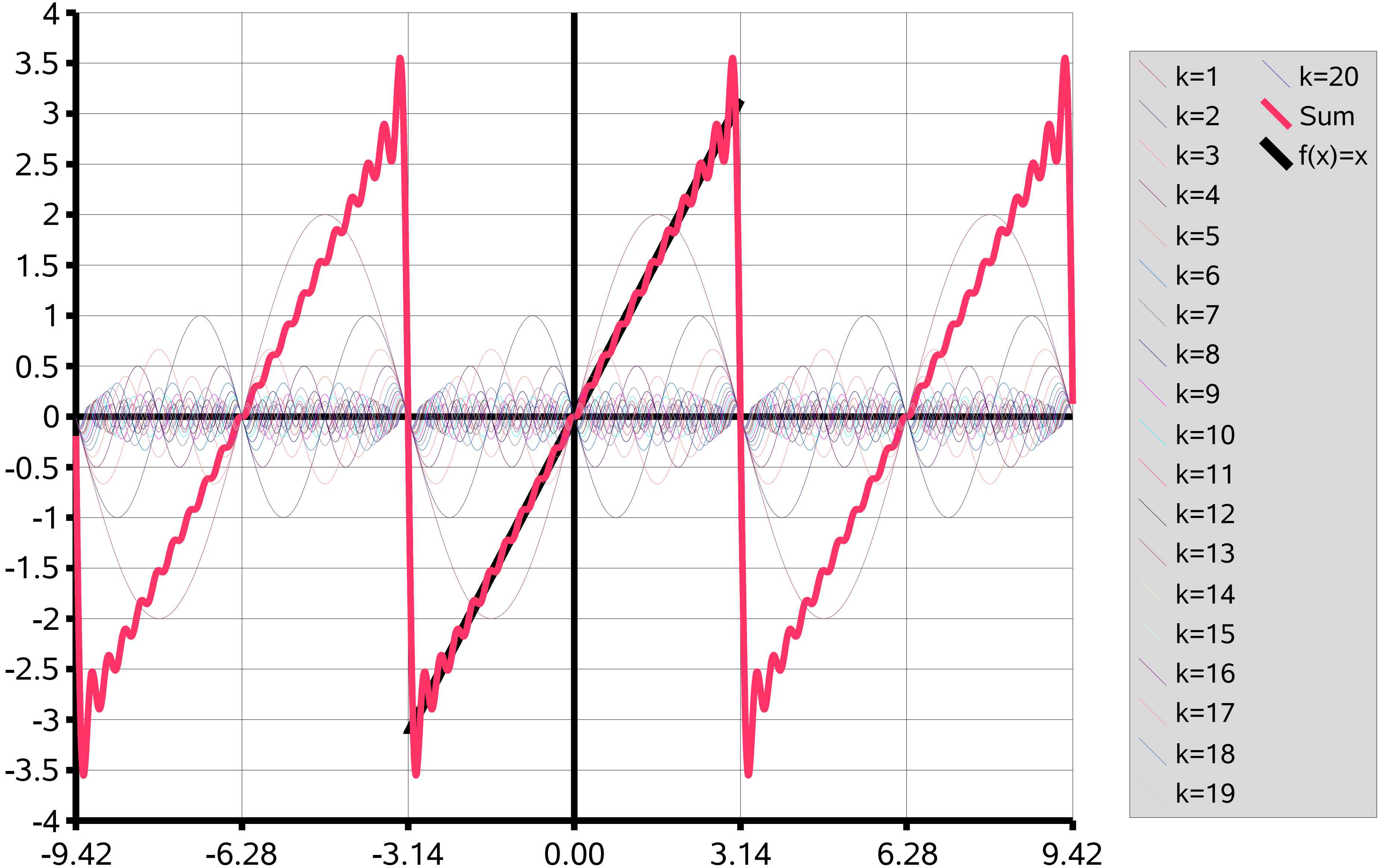
$$f(x)=\sum_{k=1}^{\infty} b_k \sin(kx)$$

$$S_{10}(x)=\sum_{k=1}^{10}\frac{2(-1)^{k+1}}{k}\sin(kx)$$



Example: Ramp

$$S_{20}(x)=\sum_{k=1}^{20}\frac{2(-1)^{k+1}}{k}\sin(kx)$$



Example: Ramp