TDEC 115 - Week 8 Recitation Problems

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New Formulas and Concepts

We are now introduced to the second part of electo*magnetics*. Magnetic fields, while in many ways similar to electric fields, are quite different in many important ways. Electric fields are produced by and act on charges, whereas magnetic fields are produced by and act on *moving* charges only.

Units and Constants

We have a new constant, μ_0, the so-called permeability of free space. It functions as a coupling constant in the strength of magnetic interactions (just like ϵ_0 for electrostatic and G for gravitational). It's value is conveniently expressed as

$$\mu_0 = 4\pi \times 10^{-7} \mathrm{T} \cdot \mathrm{m/A}.$$

We also have one new unit, the tesla T, where

$$[T] = 1\mathrm{N} \cdot \mathrm{A/m}.$$

Cross Product Rules

Remember the alphabetical order $i\ j\ k\ i\ j$ and that you get a $+$ going right and $-$ going left. So $i \times j = k$ (going right), but $j \times i = -k$ (going left).

$$\begin{array}{c} \longrightarrow + \\ ijkij. \\ - \longleftarrow \end{array}$$

Right Hand Rule

The right-hand-rule gives you the correct direction when figuring out a cross-product. If you have a formula like $\vec{A} \times \vec{B} = \vec{C}$, then pointing your first two fingers the in the directions of the first two vectors, your third finger will point in the direction of the third vector (answer). But you must do it in order! So $\vec{A} \rightarrow$ thumb, $\vec{B} \rightarrow$ index finger, so the result $\vec{C} \rightarrow$ middle finger. Just make sure you use your *right* hand!

Force on a moving point charge

$$\vec{F} = q\vec{v} \times \vec{B}.$$

Force on a current carrying wire

$$d\vec{F} = Id\vec{L} \times \vec{B}.$$

Note that if $\vec{B}$ is constant then this expression can be integrated immediately to get $\vec{F} = I\vec{L} \times \vec{B}$, which is the form in which we will usually use it. $\vec{L}$ has length equal to the physics length of the wire and points in the current direction.

Note that for a closed circuit in a constant a field we have

$$\begin{aligned} \vec{F} &= \int_i^f Id\vec{L} \times \vec{B} \\ &= I\left(\int_i^f d\vec{L}\right) \times \vec{B} \\ &= I\left(\vec{L_f} - \vec{L_i}\right) \times \vec{B} \\ &= 0, \end{aligned}$$

where the last line follows since a closed loop ends up where it starts.

Work

Also, note that the work done by the magnetic force is given by

$$\begin{aligned} W &= \int \vec{F} \cdot d\vec{r} \\ &= \int \left(q\vec{v} \times \vec{B}\right) \cdot \vec{v}dt \\ &= 0, \end{aligned}$$

because the cross product of two vectors is *always* perpendicular to the two you started with. In particular, $\vec{v} \times \vec{B}$ is perpendicular to $\vec{v}$, so their dot product must give zero. This result is fully general as we made no assumptions regarding v or B. Thus, the magnetic field *never* does any work. Though it may change the *direction* of velocity, it never changes the *magnitude*. In particular, for constant B problems, we will only have combinations of straight and circular motion (for example, the helix problems Venkat loves so much...)

Solutions

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This is a very good, but rather difficult problem. We want to solve for the velocity as a function of time, so we're looking to solve the equation of motion.

But, before we do it analytically, lets think about the solution. We know the magnetic field does not act on a component of velocity parallel to it (remember the cross product?). So, the x-component here will remain unchanged. Now, the y-component is now perpendicular to B. But, the magnetic field can only change its direction.

Now the right hand rule says that force is upward. So the y-component begins to rotate upward. But, as it rotates, the force changes so as always to be perpendicular to the velocity direction. In other words, this is a problem of centripetal acceleration! The direction of the perpendicular component will go around in a circle, but with always the same magnitude.

Now, put the two together. The *parallel* component of velocity remains unaltered, while the *perpendicular* component goes around in a circle - the path should be that of a helix, or spring. In fact, if we continue our analysis along these lines we can get the precise mathematical form of the answer. Try it and see if you get what is derived below through some labor.

Now, lets look at this analytically. We want to solve the equations of motion, so we have Newton's equation $\vec{F} = m\vec{a}$, with the magnetic force on the left-hand-side. Thus we have

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}(t)}{dt} = q\vec{v}(t) \times \vec{B},$$

where the complication arises because v is on both sides - we have a differential equation.

But, we can expand out v in terms of it's components and take the cross product with B as follows:

$$\begin{aligned} \vec{v} \times \vec{B} &= (v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}) \times (B\hat{\imath}) \\ &= -Bv_y\hat{k} + Bv_z\hat{\jmath}, \end{aligned}$$

which follows using the rules above (note that the t's have been dropped for ease of notation). Now using this expression, Newton's equation becomes

$$\frac{d\vec{v}}{dt} = \frac{qB}{m}(v_z\hat{\jmath} - v_y\hat{k}).$$

This is a vector equation, which stands for 3 separate equations when written out in components:

$$\begin{aligned} \frac{dv_x}{dt} &= 0 \\ \frac{dv_y}{dt} &= \omega v_z \\ \frac{dv_z}{dt} &= \omega v_y, \end{aligned}$$

where we have set $\omega \equiv qB/m$ for convenience. The first equation is easy - no acceleration in x. The x-velocity never changes so $v_x = v_{x0}$ for all time.

The second two equations, however, are rather nasty - coupled differential equations. They are coupled because the *change* in one depends on the current *value* of the other. However, it is actually not too difficult to work this out. Treat it like any other system of equations with two unknowns - try to eliminate variables and then solve for the unknowns. For instance, the second equation gives

$$v_y = \frac{-1}{\omega}\frac{dv_z}{dt}.$$

We can then substitute this expression into the third equation to obtain

$$\frac{d^2v_y}{dt^2} = -\omega^2 v_y.$$

If we do the same procedure with v_z we get the same thing but with y replaced by z

$$\frac{d^2v_z}{dt^2} = -\omega^2 v_z.$$

Voila! The equations have been decoupled. Now, we need to solve these two (identicle) equations - we need functions that, after differentiated twice, give the the same thing back but with a minus sign. A little though or intuition will yield $\sin(\omega t)$ and $\cos(\omega t)$ (you can check these by differentiation).

Now, which do we use? Or do we use both? Here is where the initial conditions come in. Our initial conditions are

$$\begin{aligned}
v_y(0) &= v_{y0} \\
v_z(0) &= 0 \\
\frac{dv_y}{dt}(0) &= \omega v_z = 0 \\
\frac{dv_z}{dt}(0) &= -\omega v_y = -\omega v_{yo}.
\end{aligned}$$

Thus, the equation for v_y should start at its maximum value (first and third equations) and should just be given by cos. Likewise, for v_z we start at zero, but have maximal, negative, derivative, which means $-\sin$. Thus

$$\begin{aligned}
v_y(t) &= v_{y0}\cos(\omega t) \\
v_z(t) &= -v_{y0}\sin(\omega t)
\end{aligned}$$

You can verify by substitution that these satisfy the coupled differential equations and our initial conditions exactly. Our full solution is therefore

$$\vec{v}(t) = v_{x0}\hat{i} + v_{y0}(\cos(\omega t)\hat{j} - \sin(\omega t)\hat{k})$$

which describes a helical path as advertised (and $\omega \equiv qB/m$ is the cyclotron frequency). Finally, given $\vec{v}(t)$ above, can you find $\vec{r}(t)$?