TDEC 115 - Week 7 Recitation Problems

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Formulas and Concepts

Now we expand our discussion of circuits by including the capacitor into the mix. To write down our circuit equations we need to know how to handle a capacitor. A capacitor has a voltage across its plates equal to $V_C = Q/C$. Now we need to know which sign to use. If the current flows into a plate, that plate should accumulate positive charge, and so should have the higher potential. Thus we have

$-Q/C$ If you go through a capacitor in the direction of the current you go *down* in potential by Q/C.

$+Q/C$ If you go through a capacitor opposite the current you go *up* in potential by Q/C.

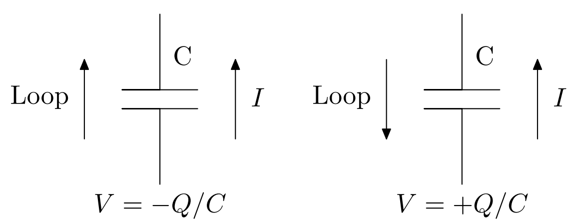

Figure 1: Voltage rules for capacitors in a circuit.

This next key idea has to do with the short ($t = 0$) and long ($t \to \infty$) time behavior of a capacitor in a circuit.

1. An uncharged capacitor acts like a *short* circuit since $V = Q/C = 0$
2. A fully charged capacitor acts like an *open* circuit since $I = dQ/dt = 0$.

Thus the short and long time problems reduce to the ones earlier involving only batteries and resisters since the capacitor no longer explicitly appears. We will typically utilize these statements by first writing down the general loop and junction equations, but then simplifying them with Q or I set to 0.

Next, we note the solutions to the differential equations for simple RC circuits. For charging we have the loop equation

$$\mathscr{E} - \frac{Q}{C} - R\frac{dQ}{dt} = 0,$$

where we have set $I = dQ/dt$, which has solution

$$Q(t) = C\mathscr{E}\left(1 - e^{-t/RC}\right),$$

where $Q_{max} = C\mathscr{E}$ is the maximum charge on the capacitor (at $t \to \infty$) and $\tau = RC$ is the time constant, whose value determines how long the capacitor takes to charge.

For discharging we have the loop equation

$$-\frac{Q}{C} - R\frac{dQ}{dt} = 0$$

where we have set $I = -dQ/dt$, which has solution

$$Q(t) = Q_0 e^{-t/RC}$$

where Q_0 is the initial charge value. Note that the current is found in both cases by differentiation. Know also what the graphs of these exponential functions look like. This is important!

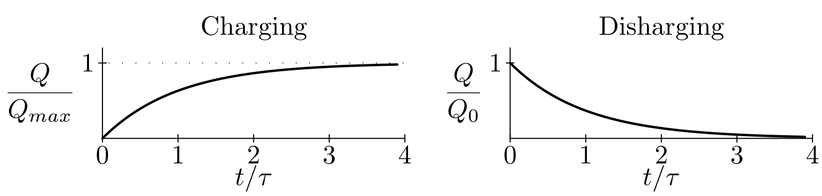

Figure 2: Graphs of the exponential functions in dimensionless quantities.

Finally, if it is needed, when you have a complicated RC circuit (one with many resisters), we can still solve the equations by writing down the $Q(t)$ equation, but making making the following replacements: $\mathscr{E}$ becomes the maximum total EMF seen by the capacitor (ie, the voltage drop across the branch with the capacitor when the current there is zero); and R becomes the total resistance in series with the capacitor (ie, mentally remove all EMFs and reduce the resulting resister network to just one in series with C). We will justify this with an example:

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I will start with the junction and two loop (left and right) equations:

$$I_1 - I_2 - I_3 \quad = \quad 0$$

$$\begin{aligned}\mathscr{E} - R_1 I_1 - R_2 I_2 &= 0 \\ -R_3 I_3 - \frac{Q}{C} + R_2 I_2 &= 0.\end{aligned}$$

Note that I_3 is the current through the capacitor, so

$$I_3 = \frac{dQ}{dt}.$$

Thus, we want to end up with a equation involving only Q and I_3. Thus we solve the first equation for I_1 and substitute into the second equation to get

$$\mathscr{E} - R_1(I_2 + I_3) - R_2 I_2 = 0,$$

which we then solve for I_2 to get

$$I_2 = \frac{\mathscr{E} - R_1 I_3}{R_1 + R_2}.$$

Now we substitute this into the third equation to remove I_2, which, after collecting terms, becomes

$$-\left(R_3 + \frac{R_1 R_2}{R_1 + R_2}\right) I_3 - \frac{Q}{C} + \frac{R_2 \mathscr{E}}{R_1 + R_2} = 0.$$

At first glance this looks entirely intractable, but notice that everything besides I_3 and Q is a constant! So, let's turn them into the more simplistic (and suggestive) constants:

$$\begin{aligned}\left(R_3 + \frac{R_1 R_2}{R_1 + R_2}\right) &\to R' \\ \frac{R_2 \mathscr{E}}{R_1 + R_2} &\to \mathscr{E}',\end{aligned}$$

which yields the equation

$$-R' \frac{dQ}{dt} - \frac{Q}{C} + \mathscr{E}' = 0,$$

which should look very familiar. Since this is *exactly* the usual RC loop equation, we immediately obtain the solution

$$Q(t) = C\mathscr{E}' \left(1 - e^{-t/R'C}\right),$$

in terms of our primed quantities. Thus the capacitor sees an effective EMF given by $\mathscr{E}'$, and this arrangement is known as a voltage divider (something you'll see a lot of in a circuits class...); and the capacitor sees an effective resistance given by R', which is written as R_3 in parallel with the series combination R_1 and R_2, which is the equivalent resistance *if we remove the EMF*! Thus, this is the justification for the statements made earlier.

Finally, let us find the voltage V_2. This quantity is the same as the voltage through the parallel branch that includes the capacitor, thus we have

$$\begin{aligned} V_2 &= V_3 + V_C \\ &= I_3 R_3 + \frac{Q}{C}, \end{aligned}$$

and we have

$$I_3 = \frac{dQ}{dt} = \frac{\mathscr{E}'}{R'} e^{-t/R'C},$$

so putting things together we obtain

$$\begin{aligned} V_2(t) &= \frac{R_3}{R'} \mathscr{E}' \left(e^{-t/R'C} \right) + \mathscr{E}' \left(1 - e^{-t/R'C} \right) \\ &= \mathscr{E}' \left[1 - \left(1 - \frac{R_3}{R'} \right) e^{-t/R'C} \right], \end{aligned}$$

Note that $R_3 < R'$ so the term in the parenthesis is positive, and the function increaes in time as we know it must. Now, if we substitute in our values we have $\mathscr{E}' = \frac{1}{2}\mathscr{E} = 600\text{V}$ and $R/R' = 2/3$, and thus

$$V_2(t) = (600\text{V}) \left(1 - \frac{1}{3} e^{-t/\tau} \right),$$

which yields the limiting values

t	$V_2(t)$
0	400V
∞	600V

Moreover, the functional form of $V_2(t)$ has exponential dependence, just as we assumed.