TDEC 115 - Week 8

Daniel J. Cross

November 16, 2005

1 Chain Rule

It is very common in physics to have accelerations given as functions of variables other than time, like position or velocity. This is because accelerations are related to force by Newton's Law $\vec{F} = m\vec{a}$, and most forces are most naturally expressed as functions of position – gravity gets weaker the farther you get from a massive body, the tension in a spring increase as you pull it away from equilibrium. And sometimes we know forces as functions of velocity as in wind resistance – the faster you go the harder the wind pushes against you.

In each of these cases we do not have acceleration as a function of time, and thus we need to know how to handle these situations. And, as in most cases, we can write down the position dependence easily, but writing down the time-dependence will be very difficult or even impossible (in closed form – we can always make numerical approximations).

So, say we're given acceleration as a function of position $a = a(x)$ and we want to find the velocity v. We have the relationship

$$a = \frac{dv}{dt}$$

so we may attempt to rearrange this equation and solve:

$$\int dv = \int a(x)dt$$

but we cannot do this because we do not know a as a function of t – this expression *cannot* be integrated!

In order to make progress we have to deal with the dependence a has on x. What we notice is that if we can express a as a function of x, then naturally we can express v as a function of x as well. Thus we have:

$$a(x) = \frac{d}{dt}\left(v\left(x\right)\right)$$

and this is where the chain rule comes in. We need to take the time derivative, but we have to go through x first. And this is perfectly acceptable as x is definitely a function of time t.

So, by the definition of derivative for any function f we have:

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(x(t+\Delta t)) - f(x(t))}{\Delta t}$$

But we have the relation

$$\Delta x = x(t + \Delta t) - x(t)$$

So that we can rewrite our derivative as

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(x+\Delta x)) - f(x)}{\Delta t}$$

And if we multiply by $1{=}\Delta x/\Delta x$ we get

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(x+\Delta x)) - f(x)}{\Delta x} \frac{\Delta x}{\Delta t}$$

So that the first fraction looks like the limit definition of the derivative of f with respect to x, while the second fraction is the derivative of x with respect to t. And since $\Delta x \to 0$ as $\Delta t \to 0$, this is precisely what we have, and the final result is

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

Now, in the current case we have $f = v$, and we can identify dx/dt as the velocity v of the particle so that we have

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

and now we can certainly separate our variables and integrate for whatever particular $a(x)$ we have:

$$\int a(x)dx = \int vdv$$

Note, this formula is particularly easy to remember because you can just think of putting in a dx/dx into the normal derivative formula. This can also be extended to functions of several variables in a straightforward way, but that would lead us too far astray.

2 An Application of the Chain Rule and the Importance of Graphs - TSV 4.35

Here we are given a as a function of x:

$$a(x) = 24 - 6x^2$$

Applying the chain rule we have

$$\begin{aligned} \int a(x)dx &= \int vdv \\ \int (24 - 6x^2)dx &= \int vdv \\ 24x - 2x^3 &= \frac{v^2}{2} + C \end{aligned}$$

If we apply our initial condition we find that $C = 24x_0 - 2x_0^3 - v_0^2/2 = 0$ and so we have

$$v = \pm\sqrt{48x - 4x^3}$$

Thus we have multiple solutions for each of the remaining questions:

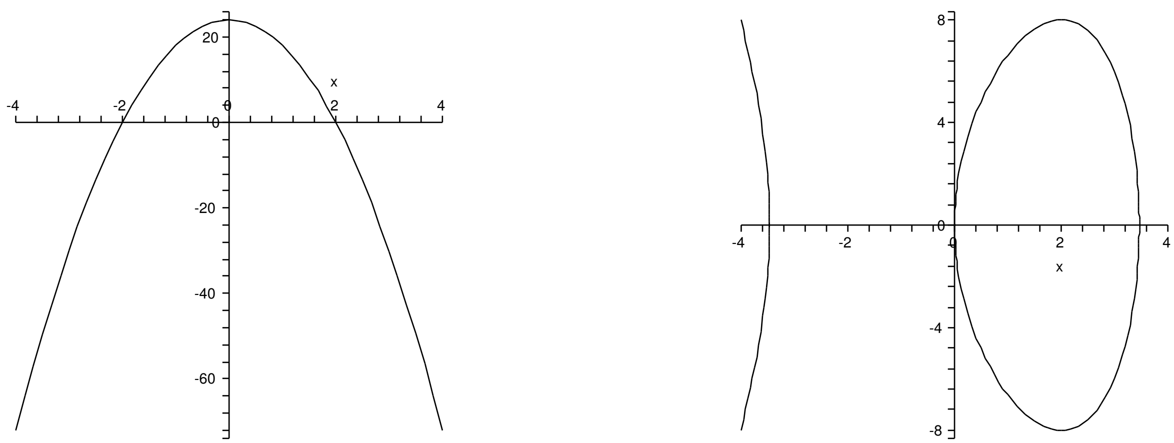
$$v(x = 2.5) = \pm 7.58\text{m/s}^2$$

$$v = 0 \quad \text{when} \quad x = 0, \pm 2\sqrt{3}\text{in}$$

$$v \quad \text{maximum when} \quad x = \pm 2\text{in}$$

Where in the last one we find the maximum by $a(x) = 0$ and solving for x.

However, we shall see that not all the solutions are applicable by looking at some graphs.

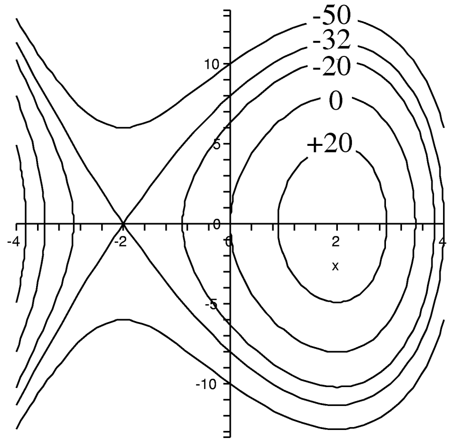


$a(x) = 24 - 6x^2$ $\qquad\qquad$ $v(x) = \pm\sqrt{48x - 4x^3}$

Figure 1: Acceleration and Velocity as functions of x

Here are a and v as functions of x for the initial condition specified ($x = 0$ when $v = 0$). What we must notice is that once we start at the origin, we will always remain on this oval part on the right – we cannot ever end up on that part on the left – they are disconnected. We say that this motion is *periodic* – after a definate amount of time we have returned to where we started. Thus we are stuck to the x values $0 \leq x \leq 2\sqrt{3}$. Thus we must reject the solution $x = -2\sqrt{3}$in for when $v = 0$, as this position is not accessible. Likewise we reject $x = -2$ for a position when v is extremal.

Now, does this mean we can never have $x < 0$? Of course we can, but that *requires a different initial condition!* This next graph shows several velocity plots for different values of the initial value parameter C, as we will see on the next page.



$$v(x) = \pm\sqrt{48x - 4x^3 + 2C}$$

Figure 2: Velocity family, C = -50, -32, -20, 0, +20

Here, a positive C value means $v = 0$ at some $x > 0$ (see the equation for C above to convince yourself of this). Thus the allowed region shrinks away from the origin (we cannot now have $x = 0$ because then $v^2 < 0$ which is impossible).

Conversely, a negative C means we have some positive v at $x = 0$. We can increase this value of v_0, and move through a region of *periodic* motion where we can have $x < 0$, until we see the "$\times$" region in the graph at $x = -2$ (this is for $C = -32$). The motion on this curve is no longer periodic since we have both $v = 0$ and $a = 0$ at the point $x = -2$. What happens is that as we approach this point we continually slow down, but never reach it as t $\rightarrow \infty$. The motion is *asymptotic*.

Finally, if we decrease C beyond -32 our two regions connect and the behavior is quite different. Since we always have $a < 0$ when $x < -2$, then if we're on the top velocity branch, our velocity decreases and we eventually get pushed toward the right half. However, once we're on the bottom half the velocity becomes more negative and we get pushed away, infinitely far to the left. Thus our motion is now *unbounded*.

One last note – this is a problem where we cannot find $x(t)$ analytically. We have

$$\frac{dx}{dt} = v(x) = \pm\sqrt{48x - 4x^3 + 2C}$$

and upon separating our variables we obtain

$$\int dt = \pm \int \frac{dx}{\sqrt{48x - 4x^3 + 2C}}$$

and at this point we are stuck - there is *no* expression for this integral in terms of elementary functions (nor even non-elementary ones, so far as I and Maple are aware), except for possibly some particular C values. So we cannot even write down $t(x)$, let alone invert it for $x(t)$.