Energy I - Week 8 Recitation Problems

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Introduction

First, let us recall what's going on here. SR is about how different inertial observers are related, that is, observers moving at constant speed with respect to each other. We will always take one frame to be 'at rest' and the other frames to be 'moving' in our calculations. This is extremely important because the answers to questions like 'where or when did x happen' depend on the state of motion of the observer.

Remember, even though we're looking at multiple reference frames drawn on the sheet of paper, we have to imagine ourselves being only one of them at any one time - usually the one at rest with respect to us (ie, our own reference frame). The most complications arise in SR when we try to imagine ourselves in two different frames at once and hitting contradictions. As long as we remember we can only be in one frame at a time, we'll do fine (nothing can be moving and not moving at the same time. If we attempt to imagine ourselves in this absurdity, no wonder we find other absurdities waiting for us).

Space-Time Diagrams

We will begin with a basic introduction to space-time diagrams, which are very useful constructions in relativity theory. Nothing keeps your thinking clearer and cleaner than a good diagram, and that is especially true in SR when things are much more complicated to begin with.

We'll build up our diagrams by first making an analogy. Consider two dimensional space. We can draw axis in the plane in different ways: say x and y (S frame) or x' and y' (S' frame) (see fig.1).

These two axis are related by a rotation by an angle θ. If we do the appropriate geometry we see that the two are related by the following transformation:

$$
\begin{aligned}
x' &= x\cos(\theta) + y\sin(\theta) \\
y' &= y\cos(\theta) - x\sin(\theta).
\end{aligned}
$$

What these equations mean is that given any point p in the plane with coordinates in S given by (x, y), we can find the corresponding coordinates of

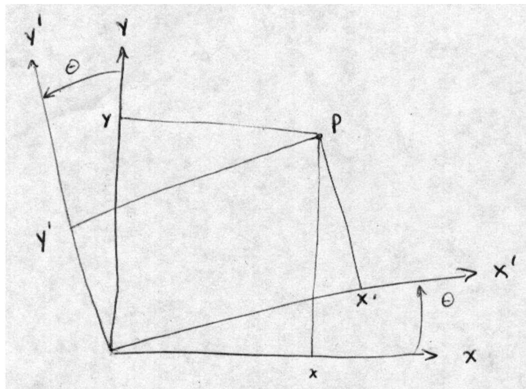

Figure 1: xy-plane with 2 different coordinate systems.

the same point in S' given by (x', y'). Depending on our coordinate system, the same point can have different coordinates. We've known this since we started doing vector analysis: the same vector will have different components in different coordinate systems, but the length of the vector is the same in all of them.

In relativity, space and time are connected similarly to how the two dimensions of the plane were connected. If an observer is moving, her coordinate system is different, so she assigns different values of position and time to events than someone who is not moving.

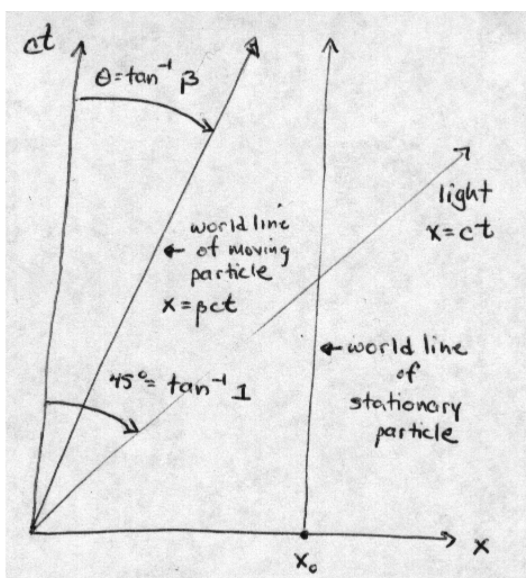

Figure 2: Basic space-time diagram.

We can now draw a new diagram (fig.2) like the last one for this new situation. We begin by making our axis to be x and ct, where c is the speed of light (so that the time axis is measured as a length - the reason for this is forthcoming).

Note that we are accustomed to having time horizontal and space vertical when we draw graphs. But, space-time diagrams are always drawn the other way around. We just have to get used to this.

Now, any point on this graph is an event - it is a moment in space and in time. How about lines? If we let the slope of the line by measured from the ct-axis, its value will be

$$\tan(\theta) = \text{slope} = \frac{\text{opp}}{\text{adj}} = \frac{\Delta x}{c\Delta t} = \frac{v}{c} = \beta,$$

that is, the slope of the line is a speed. Thus we can interpret a straight line with slope β from the $y-$axis as something moving with that constant speed. Moreover, the slope gives us the tangent of the angle made with the y axis, $\beta = \tan\theta$.

What if it moves at the speed of light? Then the slope is $\beta = 1$ and we have a line of slope 1, that is, at $45°$. This is why we scale the time axis by c - to get the light rays to appear as $45°$ lines. Light lines are of central importance in SR (they are invariants) so we want their representation to be as simple as possible. Note now that any physical motion must have $\beta < 1$, so that the line representing that motion is always sloped less with respect to the y-axis than $45°$.

Now, since a sloped line represents the motion of another observer, say, that's the next part of the diagram. That line is the time axis ct' of the moving observer. What about the x' axis? Well, remember that Lorentz transformations preserve the speed of light, so we must have

$$c = \frac{\Delta x}{\Delta t} = \frac{\Delta x'}{\Delta t'},$$

which means that if the t' axis is rotated to the right by θ, then the x' axis is rotated *up* by θ! (See fig.3).

Now we can represent two (or more) observers on the same graph. We answer all of our questions by translating between coordinates in S and coordinates in S' by using the Lorentz transformation equations:

$$\begin{aligned} x' &= \gamma(x - \beta ct) \\ ct' &= \gamma(ct - \beta x). \end{aligned}$$

What we have done so far is build a little machinery to make SR calculations easier. Learning new machinery is difficult for 2 reasons. First, it's something new to learn and we might now know how to apply it. Second, we don't know how this will actually make our lives easier - it's just excess baggage. Well, we'll use these diagrams to solve most of the problems in this chapter and through these examples we'll gain an appreciation of how these help clarify the problems at hand.

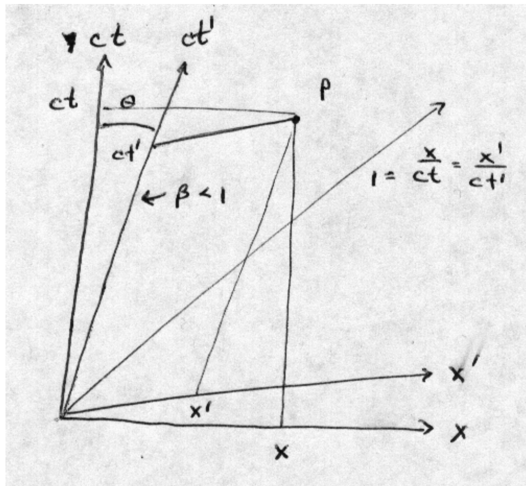

Figure 3: Space-time diagram with two reference systems.

36-4

In this problem, the father wants take a very speedy trip, so that his daughter left on earth will age 40 more years than he does. But he ages 4 years during the trip, so she must age 44 years total. More explicitly, if we let n be the daughters initial age and m her final age, we have

age	d	f
i)	n	$n+20$
f)	m	$m-20$

The we have $(m-20)-(n+20)=4$, that is, the father ages 4 years. Simplifying the expression yields $m-n=44$, so that the daughter ages 44 years.

On the diagram below (fig.4) the daughter stays on earth, so her line is just the ct axis. She is at $x=0$ at every time t. The father moves with constant speed away from earth for 2 years in his frame ($t'=2$yr) and then abruptly turns around and heads back to earth at the same constant speed for another 2 years (in his frame). The daughter ages 44 years in her frame during the whole trip, or 22 years on the father's way out and 22 more on his way back.

So, our event of interest is when the father turns around in his trip. We know both the t and t' coordinates of this point, so lets see if we can find the corresponding β. The Lorentz time transformation says that

$$ct'=\gamma(ct-\beta x),$$

so that we need to know x, that is, how far the daughter measures the father to have traveled before turning around. Well, if the father moves with constant

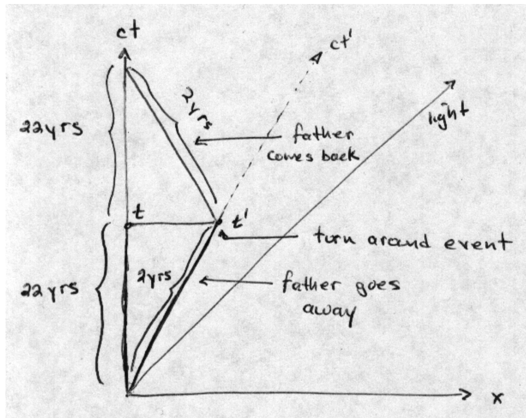

Figure 4: Space-time diagram for 37-4.

speed $v = \beta c$ and for a time t, then the distance traveled is

$$x = vt = \beta ct.$$

We could also determine this from the second Lorentz transform equation noting that the x' coordinate of when the father turns around is exactly 0 because one is always at rest in their own frame of reference (the origin of his coordinate system is always centered on him). Then we have

$$0 = x' = \gamma(x - \beta ct),$$

which immediately gives $x = \beta ct$ since $\gamma \neq 0$.

So, now we can substitute for x giving

$$\begin{aligned} ct' &= \gamma(ct - \beta(\beta ct)) \\ &= \gamma ct(1 - \beta^2) \\ &= \gamma ct(\gamma^{-2}) \\ &= ct/\gamma, \end{aligned}$$

or

$$t' = \frac{t}{\gamma},$$

which is the equation for time dilation: since $\gamma > 1$ the time t measured in the stationary frame is larger than the time t' measured in the moving frame by the factor γ. Note that we made the substitution $1 - \beta^2 = \gamma^{-2}$. (Show this is true.)

Now we just substitute in for gamma and solve for β

$$\begin{aligned} t' &= \sqrt{1 - \beta^2}t \\ (t')^2 &= (1 - \beta^2)t^2 \end{aligned}$$

$$\begin{aligned}\left(\frac{t'}{t}\right)^2 &= 1-\beta^2 \\ \beta &= \sqrt{1-\left(\frac{t'}{t}\right)^2} \\ &= \sqrt{1-\left(\frac{2}{22}\right)^2} \\ &= .996.\end{aligned}$$

37-5

In this problem we can to compare measurements made in the stationary lab frame with those 'made' by a moving particle. In the diagram below (fig.5), x is the distance traveled by the particle in the lab frame before it disintegrates, and t is the corresponding time (measured in the lab frame). Then t' is the time interval until the disintegration as measured by the particle.

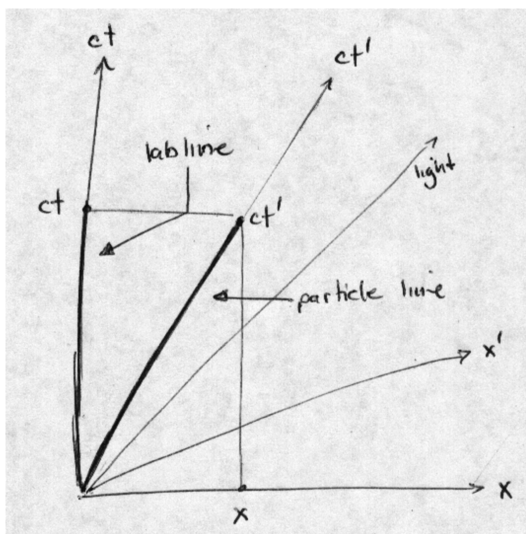

Figure 5: Space-time diagram for 37-5.

We can see that the problem is very similar to the last one. In fact, all the algebra is going to be the same and we will just quote the result that

$$ct' = \frac{ct}{\gamma}.$$

Now, we don't know t this time, but we do know x and β, the other quantities measured in the lab frame. We have

$$ct = \frac{x}{\beta} = \frac{1.05 \text{ mm}}{.992} = 1.0585 \text{ mm}.$$

We can also compute γ:

$$\gamma = \left(1 - .992^2\right)^{-1/2} = 7.9216.$$

Then we have

$$ct' = \frac{ct}{\gamma} = \frac{1.0585 \text{ mm}}{7.9216} = .13362 \text{ mm},$$

or $t' = 4.45 \times 10^{-13}$ s.

37-9

In this problem we are concerned with measurement of the length of a moving object. We measure an object by noting where the front and back of the object is *at the same time* and then find the length between these two places. When we say the 'rest length' we mean the length of an object as measure when the object is at rest, that is, how long the object judges itself to be (since anything is always at rest with respect to itself).

In the diagram (fig.6), the moving frame is that of the space ship. The ct' axis will denote the rear of the space ship (the rear is at $x' = 0$ at every time t'), and a line parallel to the ct' axis through $x' = L'$ will denote the front of the ship (the front is at $x' = L'$ at every time t'). We then note that L', the length of the ship as measured in the ship's frame, is the rest length of the ship.

Now, what we want to do is measure the length of the ship in the stationary frame. How do we do this? We pick some time, say $t = 0$, and measure where the front and back of the spaceship are at those times and take the difference. We know that at $t = 0$ the rear of the spaceship is at the origin, so $x_R = 0$ (the rear). Where's the front end? Well, we know that $x' = L'$ always, so lets see if we can use the Lorentz transformation to find out where this is in the stationary frame.

$$\begin{aligned} x' &= \gamma(x - \beta ct) \\ x' &= \gamma x \\ L' &= \gamma L, \end{aligned}$$

since we measure at $t = 0$, and L and L' are the lengths in the lab and ship frames respectively. We have $\gamma = (1 - .740^2)^{-1/2} = 1.4868$, so

$$L = \frac{L'}{\gamma} = \frac{130 \text{ m}}{1.4868} = 87.44 \text{ m}.$$

We have just deduced the relativistic length contraction formula from the space-time diagram.

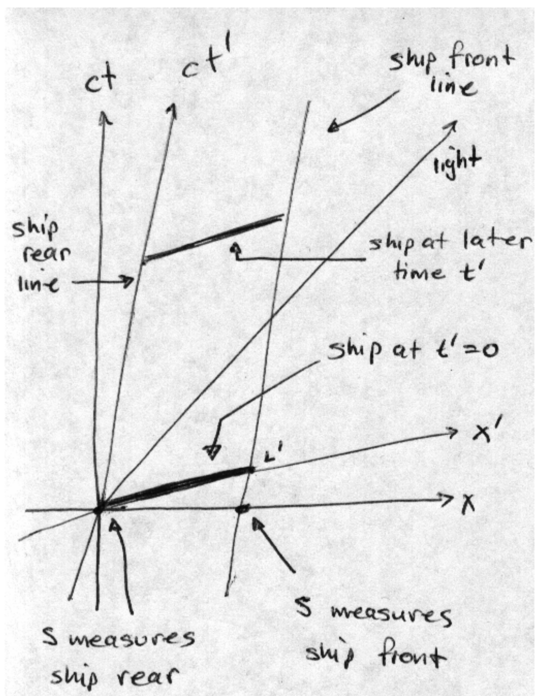

Figure 6: Space-time diagram for 37-9.

Next we want to know how long it takes for the ship to pass a given point as measured by the lab. Well, we have a measurement of how fast the ship is, and we have a measurement of how long it is (we just found it), so we can find the time by dividing:

$$ct = \frac{L}{\beta} = \frac{87.44 \text{ m}}{.740} = 118.162 \text{ m},$$

or $t = 3.94 \times 10^{-7}$ s.

37-12

This problem is a measurement problem much like the previous one, where the moving object is measured to be half as long as its rest length, that is

$$L = \frac{1}{2}L',$$

but we know that lengths are measured shorted by exactly γ, so we must have $\gamma = 2$. The corresponding β is then given by

$$\beta = \sqrt{1 - \gamma^{-2}} = \sqrt{1 - 1/4} = \sqrt{3}/2 = .8660.$$

Next, since moving clocks are judged to run more slowly by the factor γ (see problem 4), the clock here runs slower by $\gamma = 2$, that is, half as fast.

37-13

In the figure below (fig.7) the ct axis represents the earth. The line at $x = 26$ ly represents vega (it is a distance $x = 26$ ly away from earth at all times t). The line from the origin to vega is the travelers path to vega. Once at vega the traveler send a (light) signal back to earth which travels with $\beta = 1$ (always the case for light).

Since we know the speed of the traveler and the distance to vega as measured by earth we can calculate the time of travel as measured by earth by dividing as usual

$$t = \frac{x}{v} = \frac{26 \text{ ly}}{.99c} = \frac{26\ c \cdot 1 \text{ yr}}{.99c} = \frac{26 \text{ yr}}{.99} = 26.26 \text{ yr},$$

where we used the fact that a light-year is the distance traveled by light in one year, which is the speed of light times one year.

Now, the light sent by the traveler back to earth travels at the speed of light, so it takes 26 years to make the trip since it travels 26 ly. So the total time is 26 ly + 26.26 ly = 52.26 ly.

Finally, to find how long the traveler judges the trip to have taken, we use the time dilation formula from problem 4 to get

$$t' = \frac{t}{\gamma} = \sqrt{1 - \beta^2}t = \sqrt{1 - .99^2}(26 \text{ ly}) = 3.70 \text{ yr}.$$

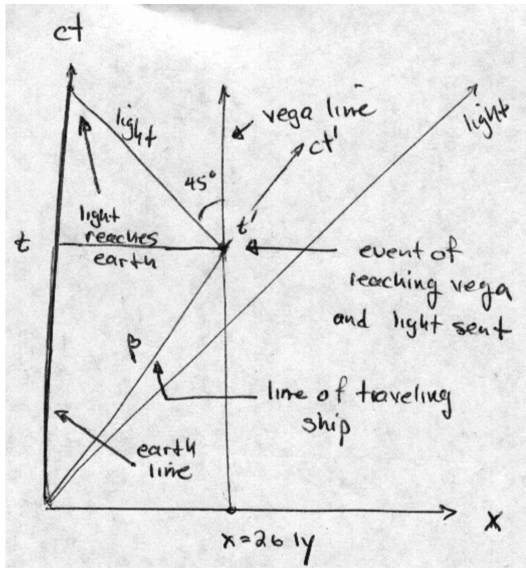

Figure 7: Space-time diagram for 37-13.

37-30

This problem discusses how the measurements of the speed of an object as made by two different observers are related. Since nothing can go faster than light as seen by any observer, we know the the relationship isn't going to be the same as in Galilean relativity. In the diagram below (fig.8) we have time axis from 3 different observers. The *ct* axis is the 'stationary' observer, S. The S' observer (ct' axis) moves with speed β_1 as measured by S, and the observer S'' (ct'' axis) moves with speed β_2 as measured in S. We will then let β_2' be the speed of S'' as measured by S'.

Note that we have labeled the angles between the time axis in the figure. We have

$$\theta_2 = \theta_1 + \theta_2'.$$

Now, remember from the section on space-time diagrams that the angle θ is related to the velocity parameter β by $\beta = \tan\theta$. So, lets take the tangent of the above equation relating the various angles and try to convert it all to β's:

$$\begin{aligned}
\theta_2 &= \theta_1 + \theta_2' \\
\tan\theta_2 &= \tan(\theta_1 + \theta_2') \\
\tan\theta_2 &= \frac{\tan\theta_1 + \tan\theta_2'}{1 - \tan\theta_1 \tan\theta_2'} \\
\beta_2 &= \frac{\beta_1 + \beta_2'}{1 - \beta_1\beta_2'},
\end{aligned}$$

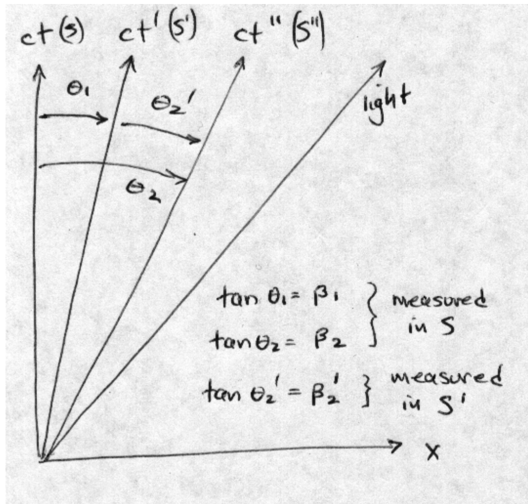

Figure 8: Space-time diagram for 37-30.

where we used a trig identity from the back of the book and in the last line substitute each $\tan\theta$ for the corresponding β. This is the formula for converting speed measurements of a third object (here S'') between two frames (S and S').

The data given is

$$\begin{aligned} \beta_1 &= .62 \\ \beta_2' &= .47. \end{aligned}$$

That is, we know the speed of the second frame as measured by the first, and the speed of the third as measured by the second. So we can use the above formula to find β_2, the speed of the third as measured by the first. We have

$$\beta_2 = \frac{.47 + .62}{1 - (.47)(.62)} = .844.$$

The classical counter part to this formula is the numerator only

$$\beta_2 = \beta_1 + \beta_2' = .47 + .62 = 1.09,$$

which gives a velocity greater than that of light. We then see that it is the denominator in our formula which ensures that no velocity can ever be measured to be greater than light.

In the next part the third object is moving in the opposite direction as seen by S', that is, $\beta_2' = -.47$ now. But he calculations are the same:

$$\beta_2 = \frac{-.47 + .62}{1 - (-.47)(.62)} = .21,$$

while the classical calculation is

$$\beta_2 = -.47 + .62 = .15,$$

and agrees more closely with the relativistic calculation since the speed is not as close to the speed of light.

37-33

This problem is very difficult without the aid of a space-time diagram to keeps things straight. In the diagram (fig.9), the S frame is the stationary space station, S' is the armada, and S'' is the messenger ship. The two events of interest are when the messenger leaves the armada (we choose the origin) and when it reaches the front of the armada. These are labeled as (1) and (2). The rear and front of the armada are the ct' axis and a line parallel to that axis through $x' = L' = 1$ ly, the rest length of the armada.

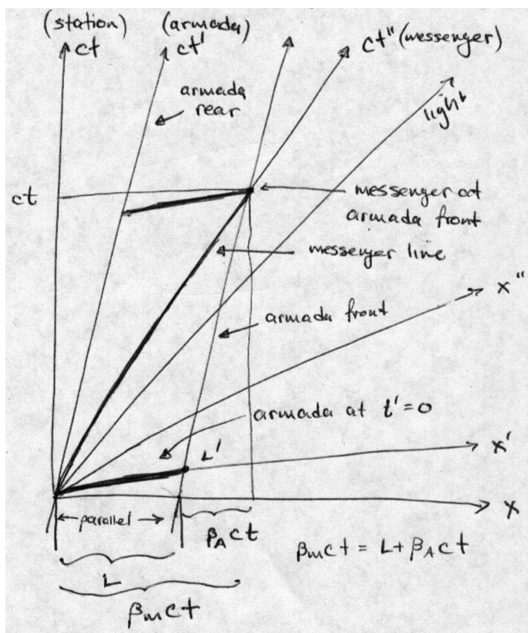

Figure 9: Space-time diagram for 37-33.

Now, what we need to calculate is the time coordinate in all three reference systems of when the messenger reaches the front of the armada. Since this is an event in space-time, all we need to do is calculate the time in one coordinate system and then convert it to the others using Lorentz transformations. I will do the calculations in the stationary S frame an then transform to the others.

As seen by S, the messenger starts at the origin and then moves with $\beta_M = .95$ until it reaches the front of the armada. But the armada is also seen to be moving with speed $\beta_A = .8$. So the ship has to move the length of the armada and move an extra distance since the armada is moving. S will judge this to take a time t. Let us denote the total distance moved as x (as judged by S). Then we have

$$x = \beta_M(ct) = L + \beta_A(ct),$$

where L is the length of the armada as judged by S. Make sure this expression makes sense to you as it is the crux of the problem.

Now that we've established that relationship we can solve for t once we know the length L of the armada as measured by S. The armada will be length contracted (it is moving as seen by S), so we have

$$L = \frac{L'}{\gamma_A},$$

where $\gamma_A = (1-\beta_A^2)^{-1/2} = (1-.8^2)^{-1/2} = 1.667$. Be careful to use the right γ's and β's! Then

$$L = \frac{1 \text{ ly}}{1.667} = .6 \text{ ly}.$$

Finally we have

$$\begin{aligned}
\beta_M(ct) &= L + \beta_A(ct) \\
ct &= \frac{L}{\beta_M - \beta_A} \\
ct &= \frac{.6 \text{ ly}}{.95 - .8} \\
ct &= 4 \text{ ly} \\
t &= 4 \text{ yr}.
\end{aligned}$$

Now we need to know the x coordinate of this event before we can transform it to the other systems. We have (substituting), that

$$x = \beta_M(ct) = (.95)(4 \text{ ly}) = 3.8 \text{ ly}.$$

So the hard work is done. Now we use the Lorentz transformations (again, being careful to use the right β's)

$$\begin{aligned}
ct' &= \gamma_A(ct - \beta_A x) \\
ct' &= 1.667(4 \text{ ly} - .8(3.8 \text{ ly})) \\
ct' &= 1.6 \text{ ly} \\
t' &= 1.6 \text{ yr},
\end{aligned}$$

and

$$\begin{aligned}
ct'' &= \gamma_M(ct - \beta_M x) \\
ct'' &= 3.2(4 \text{ ly} - .95(3.8 \text{ ly})) \\
ct'' &= 1.25 \text{ ly} \\
t'' &= 1.25 \text{ yr}.
\end{aligned}$$

As an exercise to the reader, find x' and x''. What do these mean?

37-43

OK, no diagrams for these. The kinetic energy of a particle in SR is given by

$$K = (\gamma - 1)mc^2.$$

I will briefly give a reason as to why the definition should be different. Suppose we apply a constant force to a particle. Then the work done is

$$\Delta K = W = F\Delta x.$$

The problem we have is that, since F is constant, the kinetic energy should increase proportionally to Δx. But the speed of the particle is bounded above by c. So the definition of kinetic energy as $mv^2/2$ gives a maximum value of $mc^2/2$ for the kinetic energy. Thus we need a new definition that has value 0 when $v = 0$ and increases to infinity as $v \to c$. The definition above does exactly that since $\gamma \to 1$ as $v \to 0$ and $\gamma \to \infty$ as $v \to c$.

In any case, a change in kinetic energy can be expressed as

$$\begin{aligned} \Delta K &= K_f - K_i \\ &= (\gamma_f - 1)mc^2 - (\gamma_i - 1)mc^2 \\ &= (\gamma_f - \gamma_i)mc^2, \end{aligned}$$

but since γ increases non-linearly with v, the answer will depend on the starting value of v, not just the change in v. We have

$$\begin{aligned} \gamma_i &= (1 - .18^2)^{-1/2} = 1.0166 \\ \gamma_f &= (1 - .19^2)^{-1/2} = 1.0185, \end{aligned}$$

so that

$$W = (1.0185 - 1.0166)(511 \text{ keV/c}^2)c^2 = 971 \text{ eV} \approx 1 \text{ keV},$$

since the mass of an electron is 511 keV/c^2.

for the second part we have

$$\begin{aligned} \gamma_i &= (1 - .98^2)^{-1/2} = 5.025 \\ \gamma_f &= (1 - .99^2)^{-1/2} = 7.088, \end{aligned}$$

so that

$$W = (7.088 - 5.025)(511 \text{ keV/c}^2)c^2 = 1.05 \text{ MeV},$$

which is about $10^6/10^3 = 1000$ times larger!

37-53

The mass of the aspirin is 320 mg $= 3.2 \times 10^{-4}$ kg. The energy 'equivalent' is

$$E = mc^2 = (3.2 \times 10^{-4} \text{ kg})(3 \times 10^8 \text{ m/s})^2 = 2.88 \times 10^{13} \text{ J}.$$

Now, the car goes 12.75 km for every liter of gas and each liter of gas provides 3.65×10^7 joules of energy. Thus the distance traveled for a given amount of energy is

$$\frac{\text{dist}}{\text{energy}} = \frac{12.75 \text{ km}}{\text{L}} \cdot \frac{1 \text{ L}}{3.65 \times 10^7 \text{ J}} = 3.49 \times 10^{-7} \text{ km/J}.$$

Thus the distance traveled when powered by an aspirin is

$$\begin{aligned} \text{dist} &= \frac{\text{dist}}{\text{energy}} \cdot \text{energy} \\ &= (3.49 \times 10^{-7} \text{ km/J})(2.88 \times 10^{13} \text{ J}) \\ &= 10^7 \text{ km!} \end{aligned}$$

By comparison, the circumference of the earth is 4×10^4 km, so that the aspirin would allow the car to go around the earth about 250 times before running out! Now that's some sweet gas mileage.