Energy I: Week 3 Recitation Problems

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16-27

In this problem they want you to find the velocity of a 'wave' with the equation

$$y(x,t) = (2.00 \text{ mm}) \left[(20 \text{ m}^{-1})m - (4.0 \text{ s}^{-1})t\right]^{1/2}.$$

It would be nice to know that this is in fact the equation of a wave. It's not too difficult to check, but it's actually easier if we abstract a little bit. Recall that the original solutions we had looked like

$$A \sin(kx - \omega t).$$

The current function is almost the same, it's form is

$$A\sqrt{kx - \omega t},$$

and the book is obviously claiming this is a wave. If we take what these two equations have in common, we might make the conjecture that every function of the form

$$Af(kx - \omega t),$$

is a wave, where in the first case $f = \sin()$ and in the second $f = \sqrt{()}$.

Checking whether a general f or a particular one amounts to the same thing - showing that it satisfies the wave equation. But it turns out to be computationally easier to stick to a general f. We need to show that $f(kx-\omega t)$ is a solution to

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}.$$

As is often the case, making variable substitutions makes life easier. Note that f can be though of as a function of one argument, which is $kx - \omega t$ (and this turns out to be the crucial property). We will write f as $f(\alpha(x,t))$, where $\alpha(x,t) = kx - \omega t$ (convince yourself that this is the same as our original f!). So we have

$$y(x,t) = Af(\alpha(x,t)),$$

as the function we want to plug into the wave equation. So let's start with the left:

$$\begin{aligned}\frac{\partial^2}{\partial x^2}y(x,t) &= \frac{\partial^2}{\partial x^2}Af(\alpha(x,t)) \\ &= A\frac{\partial^2}{\partial x^2}f(\alpha(x,t)) \\ &= A\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}f(\alpha(x,t))\right),\end{aligned}$$

since A is just a constant. Now we have

$$\frac{\partial}{\partial x}f(\alpha(x,t)) = \frac{d}{d\alpha}f(\alpha(x,t)) \cdot \frac{\partial\alpha(x,t)}{\partial x},$$

by the chain rule. But remember what alpha is. So we get

$$\frac{\partial\alpha(x,t)}{\partial x} = \frac{\partial}{\partial x}(kx - \omega t) = k.$$

Thus

$$\frac{\partial}{\partial x}f(\alpha(x,t)) = k\frac{d}{d\alpha}f(\alpha(x,t)),$$

which isn't too bad an expression. Now do the second derivative:

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}f(\alpha(x,t))\right) &= \frac{\partial}{\partial x}\left(k\frac{d}{d\alpha}f(\alpha(x,t))\right) \\ &= k\frac{\partial}{\partial x}\left(\frac{d}{d\alpha}f(\alpha(x,t))\right) \\ &= k\frac{d^2}{d\alpha^2}f(\alpha(x,t)) \cdot \frac{\partial\alpha(x,t)}{\partial x} \\ &= k^2\frac{d^2}{d\alpha^2}f(\alpha(x,t))\end{aligned}$$

where we used the chain rule and our expression for the x-derivative of α again. So, altogether we get

$$\frac{\partial^2}{\partial x^2}y(x,t) = Ak^2\frac{d^2}{d\alpha^2}f(\alpha(x,t)).$$

Ok, so now for the right. This is done exactly the same as the left was, only our derivatives are with respect to t instead of x.

$$\begin{aligned}\frac{\partial^2}{\partial t^2}y(x,t) &= \frac{\partial^2}{\partial t^2}Af(\alpha(x,t)) \\ &= A\frac{\partial^2}{\partial t^2}f(\alpha(x,t))\end{aligned}$$

$$
\begin{aligned}
&= A\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}f(\alpha(x,t))\right) \\
&= A\frac{\partial}{\partial t}\left(\frac{d}{d\alpha}f(\alpha(x,t))\cdot\frac{\partial\alpha(x,t)}{\partial t}\right) \\
&= A\frac{\partial}{\partial t}\left(\frac{d}{d\alpha}f(\alpha(x,t))\cdot(-\omega)\right) \\
&= -\omega A\frac{\partial}{\partial t}\left(\frac{d}{d\alpha}f(\alpha(x,t))\right) \\
&= -\omega A\frac{d^2}{d\alpha^2}f(\alpha(x,t))\cdot\frac{\partial\alpha(x,t)}{\partial t} \\
&= \omega^2 A\frac{d^2}{d\alpha^2}f(\alpha(x,t)),
\end{aligned}
$$

where the steps follow just as before, but we use

$$\frac{\partial\alpha(x,t)}{\partial t} = \frac{\partial}{\partial t}(kx-\omega t) = -\omega.$$

So, the right hand side becomes

$$\frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\omega^2 A}{v^2}\frac{d^2}{d\alpha^2}f(\alpha(x,t)),$$

and the wave equation is now

$$Ak^2\frac{d^2}{d\alpha^2}f(\alpha(x,t)) = \frac{\omega^2 A}{v^2}\frac{d^2}{d\alpha^2}f(\alpha(x,t)).$$

Now if we cancel the common factors on both sides of the equation it reduces to

$$k^2 = \frac{\omega^2}{v^2},$$

because f as completely disappeared! Thus our function IS a solution to the wave equation, and moreover we know know what the speed is - it's ω/k, just as before!

Note the importance of this result. *Any* (twice differentiable) function $f(z)$ of one variable is a solution of the wave equation if we make it a function of $kx-\omega t$. Thus any shape can be made into a propagating wave, and that shape will be traced out in space and in time, and it will move with speed $v = \omega/k$ undistorted. This is a property of the the fact that the wave equation is a *homogeneous second-order differential equation*. In materials, for example, this equation is only valid for small oscillations. If larger oscillations are needed, the wave equation picks up terms up higher order in space and ceases to be homogeneous. Then waves cannot propagate without distortion. We have refraction.

Finally, note that setting up the appropriate initial conditions to get a propagating wave on a desired shape is not a trivial problem!